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Nominal Tense Logic and Other Sorted Intensional Frameworks

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For Andrew Holster and Josh

Abstract

This thesis introduces of a system of tense logic called *nominal tense logic* (NTL), and several extensions. Its primary aim is to establish that these systems are logically interesting, and can provide useful models of natural language tense, temporal reference, and their interaction.

Languages of nominal tense logic are a simple augmentation of Priorean tense logic. They add to the familiar Priorean languages a new sort of atomic symbol, *nominals*. Like propositional variables, nominals are atomic sentences and may be freely combined with other wffs using the usual connectives. When interpreting these languages we handle the Priorean components standardly, but insist that nominals must be true at one and only one time. We can think of nominals as naming this time.

Logically, the change increases the expressive power of tensed languages. There are certain intuitions about the flow of time, such as irreflexivity, that cannot be expressed in Priorean languages; with nominals they can. The effects of this increase in expressive power on the usual model theoretic results for tensed languages discussed, and completeness and decidability results for several temporally interesting classes of frames are given. Various extensions of the basic system are also investigated and similar results are proved. In the final chapter a brief treatment of similarly referential interval based logics is presented.

As far as natural language semantics is concerned, the change is an important one. A familiar criticism of Priorean tense logic is that as it lacks any mechanism for temporal reference, it cannot provide realistic models of natural language temporal usage. Natural language tense is at least partly about referring to times, and nowadays the deictic and anaphoric properties of tense are a focus of research. The thesis presents a uniform treatment of certain temporally referring expressions such as indexicals, and simple discourse phenomena.

However the thesis also has a secondary aim: to establish the importance of sorting in intensional logic. NTL and all the other systems treated here are examples of sorted intensional languages. General logical questions concerning sorting are discussed, and the possibility of using sorted intensional languages to provide event logics is raised.

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Chapter 1

Introduction

This thesis is primarily an essay on temporal logic, and more specifically, tense logic. It introduces a system called *nominal tense logic* (NTL), and several extensions, and attempts to establish two things: that these systems are logically interesting, and that they can provide useful models of natural language tense, temporal reference, and their interaction.

Languages of nominal tense logic are a simple augmentation of Priorean tense logic. They add to the familiar Priorean languages a new sort of atomic symbol, *nominals*. We follow the usual conventions and represent ordinary propositional variables by means of p, q, r and so on; nominals are typically represented by means of i, j and k . Like propositional variables, nominals are atomic sentences. We can freely make wffs in the usual way, using both sorts of atom, out of the usual connectives \wedge, \neg, P, F, H , and G . Wffs of NTL thus look much like wffs of ordinary tense logic: typical examples are the *purely nominal* wff, $FFi \rightarrow Fi$; the *purely Priorean* wff, $Gp \rightarrow Fp$; and the *mixed* wff $F(i \wedge p)$.

When interpreting these languages we handle the Priorean component in the usual fashion, but place a constraint on the interpretation of nominals: we insist that nominals must be true at one and only one time. We can think of nominals as naming the time they are true at, and have thus added a mechanism of temporal reference to Priorean tense logic. A great part of the thesis is concerned with exploring the logical and formal semantical consequences of this simple change.

Logically, the change increases the expressive power of tensed languages. There are certain natural constraints on the flow of time, such as irreflexivity, that cannot be expressed in Priorean languages; with nominals they can. In fact, by using both nominals and variables we can pin down both the natural numbers and the integers up to isomorphism. This increase in expressive power effects all the usual model theoretic results for tensed languages — for example p-morphisms no longer preserve validity — nonetheless it is possible to give completeness and decidability results for many temporally interesting classes of frames.

As far as natural language semantics is concerned, the change is an important one. A familiar criticism of Priorean tense logic is that as it lacks any mechanism for temporal reference, it is simply incapable of providing realistic models of natural language temporal usage. As writers such as Reichenbach have argued, tense is about referring to times; and nowadays the deictic and anaphoric properties of tense are the focus of research. As we shall see, the simple devices we introduce in this thesis enable us to cope with a wide range of temporally referring expressions rather routinely.

Nominals, and languages of nominal tense logic, don't do all the work in the thesis. Once the idea of referring to instants using atomic sentences has been accepted, it seems natural to try referring to extended periods. Accordingly, *interval nominals* are introduced. These refer to intervals in just the same way that nominals refer to points of time — interval nominals are constrained in their interpretation to be true only at the set of points in some unique interval. Once systems with nominals and interval nominals have been defined it becomes natural to introduce a further operator in addition to the Priorean pair F and P . The operator is here called the shifter. Actually its just a universal modality L — it means 'true at all times' — but in combination with a nominal or an interval nominal it provides a smooth way of 'shifting' the point of evaluation to the point (or interval) named. Languages with nominals, interval nominals, and the shifter are called languages of TREF, and these are the languages in which much of our work on natural language is done.

However the thesis is not just about tense logic, but has a secondary theme: *sorted*

intensional languages. Nominals, interval nominals, and ordinary variables are all examples of sorts, and NTL and TREF are both sorted intensional languages. The thesis tries to show that the idea of sorting in intensional frameworks is logically interesting, and suggests that it has a potentially wider significance for natural language semantics than merely providing models of temporal reference.

The word ‘sort’ has been used already: nominals were described as a new sort of atom. The analogy intended with first order sorted languages should be clear: each sort embodies a different type of information. Nominals always encode ‘single point of time information’ and interval nominals encode ‘single interval information’. Knowing what sort an atom belongs to already tells us something. We may not know what it names, but we know what it can name. However although different sorts have different semantic import, the different types of information they embody are combined in syntactically uniform fashion. ‘Mixed yet uniform’ is the best description of the calculi considered here.

Logically, the interesting thing about sorting in intensional frameworks is that it does make a difference — witness the new expressive powers of nominals. Here the analogy with first order sorted languages breaks down interestingly. Sorting first order languages results in syntactic sugar; sorting intensional languages can lead to rich new frameworks.

By and large, the discussion of sorting in this thesis confines itself to nominals and interval nominals. Even so, a number of more general logical concerns emerge (the importance of classes of models rather than classes of frames for sorted logics), and a number of more general ideas for applying sorting to model more than temporal reference (for example, to model a richer notion of event structure), are mentioned with increasing frequency as the thesis progresses.

The reader is invited to read the thesis in the following way. Firstly, it should be read as an exercise in pure and applied tense logic whose function it is to establish the logical interest of NTL (and TREF) and demonstrate their applicability. But in addition it should also be read as (so to speak) an existence proof. I believe that sorting is of wider logical and applied interest than is demonstrated in this thesis, but while I suggest why

this might be so, I do not establish it here. Nonetheless, I believe that the results here show the interest (both logical and formal semantical) of at least one sort (nominals). I don't claim there must be more; I do claim it is worthwhile looking further.

The chapter by chapter plan of the thesis is as follows. The first four chapters are a leisurely and detailed exploration of the logical properties of NTL. In Chapter 2 ("Languages of Nominal Tense Logic") we define the syntax and semantics of these languages, note some of their basic properties, and define other important concepts (such as that of a filtration) which will be used throughout the thesis. In Chapter 3 ("Model Theory") we begin the task of demonstrating the logical interest of these systems. We first note a number of examples of the increase in expressive power, investigate how the increase has affected the standard tense logical preservation results, and finally compare NTL with three classical languages and with D logic. In Chapter 4 ("The Minimal Logic") we axiomatise the NTL validities by augmenting the minimal standard tense logical axiomatisation with a single schema. The resulting axiomatisation is called K_{nt} . In fact four suitable schemas are considered, and minimal nominal modal logic is examined as well. In Chapter 5 ("Extensions of K_{nt} ") we axiomatise the nominal tense logics of various classes of frames of interest. Probably the most interesting result of the chapter is the decidability results proved for certain newly definable classes of frames. (These hinge on the fact that Segerberg's Theorem fails in NTL. Possession of the finite model property does not imply possession of the finite frame property.)

After this survey, the concerns broaden. In Chapter 6 these wider concerns are logical, and in Chapter 7, formal semantical. Chapter 6 ("Interval Nominals, the Shifter, and Sorting Generalised") does three things. In the first two sections it adds, successively, interval nominals and the universal modality L , thus forming the languages of TREF. Their basic logical theory is developed, and the way is thus paved for the explorations of Chapter 7. However by this stage it will have become apparent that the sorted atomic sentence strategy gives rise to a number of new questions. The final section indicates what these new issues are, and tries to formulate a general account of sorting. (Until this stage of the thesis, while the word sort has been freely used, no real attempt has been made to pin it down.) Sorting is (tentatively) defined, and a number of possible

directions for further work are noted. This chapter concludes the main stream of logical work of the thesis.

Chapter 7 (“Applications in Natural Language Semantics”) attempts to establish the other claim of the thesis: that these logics are useful in natural language semantics. Languages of TREF are first treated as straightforward embodiments of the referential or Reichenbachian view of tense. Following this we add further constrained atoms to our languages, this time to model the effects of indexicals such as ‘now’ and ‘yesterday’ and calendar terms such as ‘Monday’. These additions are all syntactically uniform: ‘now’, and ‘Monday’ are treated simply as atomic symbols on a par with propositional variables or nominals. We show that they interact properly with our tense operators, and correctly treat such sentences as “John will run yesterday” as anomalous. The indexicals are handled semantically by a Kaplan style contextualised semantics, and the similarities and differences with other intensional approaches to these issues are noted. This is followed by a very important section. We compare our strategy with the two dimensional tradition initiated by Frank Vlach. We note van Benthem’s criticism of these two dimensional systems — basically, that they amount to rather complicated ways of doing first order logic without admitting it — and argue that it does not apply to our sorted languages. For, as is shown, we are able to cope with the type of sentences that inspired these two dimensional additions in the language of TREF, a language that is one dimensional, and both intuitively and formally simple. In a final section we note some of the features of discourse semantics that we can handle in TREF, and discuss an interesting overlap of TREF with temporal DRT.

At the conclusion of this chapter both major goals of the thesis are felt to have been achieved. A lengthy logical exploration has established the interest of these languages, and the examination of natural language semantics has demonstrated that a wide variety of phenomenon can be treated in very simple fashion by introducing appropriately constrained atoms into intensional languages. However (at least) two loose ends remain. Although in languages of TREF we can refer to intervals, we are not doing ‘interval semantics’ in the generally accepted sense of the word. (This is not felt to be a bad thing incidentally; one of the pleasant things about sorting is the number of options it opens.

Interval semantics need not mean intervallic evaluation.) What would sorted interval based languages be like? The results presented here are only preliminaries to a fuller investigation, nonetheless they are interesting. The section concludes with a brief discussion of sorting in the interval based framework. Following this, the work of Prior and Bull on systems of logic with quantified nominals is discussed.

When I began working on NTL I believed the idea of nominals be a new one. This was mistaken: both Prior and Bull had considered the idea in the 1960s. However they used nominals in a far more powerful way — nominals could be quantified over — and we defer considering their work until the end of the thesis. However there has also been recent work (predating mine) by a group of Bulgarian logicians — George Gargov, Valentin Goranko, Solomon Passy and Tinko Tinchev — who have variously co-authored papers on systems of both propositional dynamic logic [76] [33] [77] and modal logic with nominals [34] [35]. The work in dynamic logic is beyond the scope of this thesis, however their work on modal logic does overlap with mine. It is a little difficult to give a full comparison since their most comprehensive work on the topic, the recent manuscript [35], only reached me two weeks prior to the submission of this thesis.¹ However while there is some overlap, the approaches have usually been different. There is one very important idea they treat which I did not consider at all; the use of a certain ‘infinitary’ rule of inference (called COV). I discuss COV later on, and also note the differences between their minimal (modal) completeness result and mine.

Actually, the idea of nominals arose in a rather roundabout fashion in my case. The thesis began with an attempt to axiomatise the system IQ, an interval based system for natural language semantics, that employs the Kleene strong three valued semantics and a variety of indexed operators [87]. On investigating the matter it turned out that the system (or at least the ‘core’ system outlined in [88]) could be simplified down to Priorean tense logic ‘plus something else’. Trying to pin down what that something

¹Also, they do not consider tense logic, and the differences between tensed and modal languages with nominals can be considerable.

else was, and how to control it, led to the minimal NTL axiomatisation. While working on this I noted the increase in express power described in Chapter 3, and the link with contextual semantics described Chapter 7, and the thesis thereafter took a different road.

²

The idea of sorting — or at least the idea of constraining the interpretation of atoms — has also been considered before. For a start, it's the idea underlying general frames. Also, in interval based logics, constraining the interpretation of atoms is both natural and important. (Indeed I suggest in the final chapter that it's practically forced on us.) In addition Johan van Benthem [8] has considered the logics that arise when variables are constrained in their interpretations on \mathbf{R} , and explicitly noted their relevance to inferences involving the progressive. In fact this thesis is probably only original in the emphasis it gives the idea: it insists that such constraints are important if we wish to do genuine *temporal* logic, and suggests that different constraints should be syntactically reflected by introducing different sorts, which (at least as a first option) are uniformly combined.

This concludes the introduction: we are now ready to investigate nominal tense logic and other sorted intensional frameworks. ³

²Space restrictions have made it impossible to include the work on IQ here; I'll merely note that using the NTL axiomatisations given here it is routine to induce proof theories onto core IQ. Those familiar with IQ need merely note that as far as the core system is concerned, parameters are just nominals. Full details of the analysis will be released shortly as a technical report.

³The reader is advised to tuck a copy of van Benthem's "Logic of Time" [5] firmly under one arm before setting off.

Chapter 2

Languages of Nominal Tense Logic

In this chapter we assemble the tools for the first task we set ourselves in the introduction: to demonstrate that sorted intensional frameworks are logically interesting by concentrating in detail on the properties of one such system, languages of Nominal Tense Logic (NTL). We define the syntax and semantics of these languages, present some of their elementary properties, and then adapt filtration theory to the new setting. Next we discuss the standard construction known as unraveling, and introduce the important concept of a path through a frame. We conclude the chapter by discussing a number of alternative and extended systems: Nominal Modal Logic (NML); multimodal logic with nominals; languages of weak NTL and strong NTL; and first order NTL.

2.1 Syntax

By a language of Nominal Tense Logic \mathcal{L} is meant a triple $\langle LOG, VAR, NOM \rangle$ such that $card(LOG) = 6$; VAR and NOM are not both empty; and LOG , VAR , and NOM are pairwise disjoint. The elements of LOG are represented in the metalanguage by \wedge , \neg , P , F , $)$ and $($. The elements of VAR are typically represented by $p, q, r, p_1, q_1, r_1, p_2, q_2, r_2 \dots$, and are called *variables*. The elements of NOM are typically represented by $i, j, k, i_1, j_1, k_1 \dots$, and are called *nominals*. We adopt $\vee, \rightarrow, \leftrightarrow, G$ and H as the usual metalinguistic abbreviations—in particular, $G =_{def} \neg F \neg$ and $H =_{def} \neg P \neg$. \perp and \top are

introduced to represent arbitrary contradictions and tautologies of the object language under consideration. We happily blur the object/metalinguage distinction and treat the metalinguistic representations of the entities of the object language as those items themselves. In particular we frequently refer to F, P, G and H as tense operators or tenses: strictly speaking the entities denoted by these symbols are the tenses and tense operators. In this spirit we now state that G and F , and H and P , are dual pairs of operators.

Given a language \mathcal{L} , if $NOM \neq \emptyset$ and $VAR \neq \emptyset$ we say \mathcal{L} is a *mixed* language; if $VAR = \emptyset$ we say \mathcal{L} is a *purely nominal* language; and if $NOM = \emptyset$ we say \mathcal{L} is a *purely Priorean* language. Occasionally we drop the ‘purely’s. By a *language with nominals* is meant any purely nominal or mixed language.

For any language \mathcal{L} we define $ATOM_{\mathcal{L}} = VAR_{\mathcal{L}} \cup NOM_{\mathcal{L}}$. By the cardinality of a language \mathcal{L} is meant the cardinality of $ATOM_{\mathcal{L}}$; and \mathcal{L} is said to be recursive iff NOM and VAR are both recursive sets. In principle languages of any cardinality are permitted, and most results we prove go through for arbitrary languages, but by and large we are interested in ‘small’ languages; so in what follows it is tacitly assumed that we are dealing with languages with *at most* a countably infinite supply of atoms. The ‘at most’ proviso is important, because some interest attaches to languages with only a finite supply of nominals. In addition, if we say we are working with a countably infinite mixed language, we mean that both the stock of variables and the stock of nominals are countably infinite.

Definition 2.1.1 (Well formed formulas) *Given a language \mathcal{L} , $WFF_{\mathcal{L}}$ is the smallest set such that:*

1. $ATOM_{\mathcal{L}} \subseteq WFF_{\mathcal{L}}$
2. $\phi, \psi \in WFF_{\mathcal{L}} \Rightarrow (\neg\phi), (F\phi), (P\phi) \text{ and } (\phi \wedge \psi) \in WFF_{\mathcal{L}}$. □

The elements of $WFF_{\mathcal{L}}$ are referred to as ‘wffs’, ‘sentences’ or ‘formulas’; and sometimes we talk of \mathcal{L} -wffs, and so on. It is convenient to distinguish purely nominal sentences, purely Priorean sentences and mixed sentences. These are wffs whose atoms are

all nominals, all variables, or a mixture of the two respectively. We adopt an easy-going attitude towards brackets, dropping them whenever possible and adhering to the usual tightness of scope conventions.

The concepts of *degree*, *temporal depth* and *subformulas* are our most frequently used syntactic tools: results are often stated or proved (typically by induction) with their help. The following definitions apply to any language \mathcal{L} :

Definition 2.1.2 (Degree) *The degree of a wff ϕ is given by:*

1. $\deg(\phi) = 0$, for all $\phi \in \text{ATOM}_{\mathcal{L}}$
2. $\deg(\neg\phi) = \deg(F\phi) = \deg(P\phi) = 1 + \deg(\phi)$
3. $\deg(\phi \wedge \psi) = 1 + \deg(\phi) + \deg(\psi)$

□

Definition 2.1.3 (Temporal depth) *The temporal depth of a wff ϕ is given by:*

1. $td(\phi) = 0$, for all $\phi \in \text{ATOM}_{\mathcal{L}}$
2. $td(\neg\phi) = td(\phi)$
3. $td(\phi \wedge \psi) = \max\{td(\phi), td(\psi)\}$
4. $td(F\phi) = td(P\phi) = 1 + td(\phi)$

□

Definition 2.1.4 (Subformulas) *The set of subformulas of a wff ϕ is given by:*

1. $sf(\phi) = \{\phi\}$, for all $\phi \in \text{ATOM}_{\mathcal{L}}$
2. $sf(F\phi) = sf(P\phi) = sf(\neg\phi) = \{\phi\} \cup sf(\phi)$
3. $sf(\phi \wedge \psi) = \{\phi, \psi\} \cup sf(\phi) \cup sf(\psi)$

□

The *proper subformulas* of ϕ are the elements of $sf(\phi) \setminus \{\phi\}$. When we say a set of \mathcal{L} -wffs Σ is *closed under subformulas* we mean that for all $\sigma \in \Sigma$, if $\sigma' \in sf(\sigma)$ then $\sigma' \in \Sigma$; that is, Σ contains all the subformulas of all its elements. This notion is important when we discuss filtrations. We say that a wff ϕ occurs in a wff ψ iff ϕ is a subformula of ψ .

The following auxiliary machinery is occasionally helpful; their inductive specification is left to the reader. We sometimes talk of the *mirror image* of a formula ϕ . This is the formula $mi(\phi)$ obtained from ϕ by uniformly replacing every occurrence of P in ϕ by F , every occurrence of H by G , and vice versa. For example, $mi(FP(i \wedge p) \rightarrow HGP)$ is the formula $PF(i \wedge p) \rightarrow GHp$, and $mi(i \rightarrow p)$ is $i \rightarrow p$. Occasionally it is convenient to think of mi as operating on arbitrary strings of symbols of a language \mathcal{L} , not just the \mathcal{L} -wffs — for example, we want to say that $mi(PGHFFP) = FHGPPF$ — and we assume that mi has been defined this way. It is useful to have a notation for tense iteration. By $F^n\phi$ is meant the formula ϕ preceded by n occurrences of the F operator. $F^0\phi$ is just ϕ . We assume analogous conventions for the P , G and H operators.

2.2 Semantics

As in standard tense logic, the most fundamental semantic structures are *frames*. Intuitively, frames are a very simple conception of temporal flow: points of time in some order.

Definition 2.2.1 (Frames) *By a frame \mathbf{T} is meant an ordered pair $\langle T, < \rangle$ such that $T \neq \emptyset$ and $< \subseteq T^2$. T is called the carrier set of the frame. The elements of a frame are usually referred to as points.* \square

Three frames we will frequently use are $\mathbf{Z} = \langle \mathbb{Z}, < \rangle$, the integers under the usual ordering; $\mathbf{N} = \langle \mathbb{N}, < \rangle$, the natural numbers in the usual order; and $\mathbf{Q} = \langle \mathbb{Q}, < \rangle$, the rationals in their usual ordering. We frequently say frames are transitive or reflexive or have some other relational property; by this is meant that its relation has this property.

Such conditions on frame orderings are discussed in detail in the next chapter. By the cardinality of a frame we mean the cardinality of its underlying set.¹

As in standard tense logic, our languages are linked with frames by means of *valuations*, functions assigning subsets of the frame to variables. The same is done in arbitrary languages of NTL, except that we put a constraint on the interpretation of nominals: nominals must denote *singleton* subsets of the frame.

Definition 2.2.2 (NTL valuations) *By a NTL valuation V for a language \mathcal{L} on a frame \mathbf{T} is meant a function*

$$V : \text{ATOM}_{\mathcal{L}} \longrightarrow \text{Pow}(T)$$

such that $V(i)$ is a singleton subset of T , for all $i \in \text{NOM}_{\mathcal{L}}$. Here, of course, T is the carrier set of \mathbf{T} . The class of NTL valuations for \mathcal{L} on \mathbf{T} is denoted by $\text{Val}_{\mathcal{L}}(\mathbf{T})$. \square

This is the point: nominals are so called because they name. They refer uniquely to points of time. As nominals are sentences, not terms, they name by taking the value true at one and only one point of time in any valuation. Nominals are, if you like, instantaneous propositions—and the instant at which a nominal is true is the instant it names.

Note that it is *not* required that every point be named by some nominal. For example if our frame was \mathbf{N} , then the valuation which assigned every nominal $\{4\}$ would be a perfectly acceptable NTL valuation. In this valuation only 4 is named. We later consider what happens if we insist that every point be named; languages obeying this extra constraint we call *strong languages* of NTL. In what follows we call NTL valuations simply valuations, only adding the NTL prefix if emphasis or contrast is needed.

¹Frames will be denoted by letters in bold font — $\mathbf{T}, \mathbf{T}', \mathbf{T}_4, \mathbf{S}''$ etc. Their carrier sets will be referred to by the same symbol in mathematical font — T, T', T_4, S'' etc. If a frame, say \mathbf{T}_1 , has been referred to in discussion, subsequent uses of T_1 and $<_1$ may safely be taken as referring to the carrier set of the frame and the frame's relation respectively. Similarly, if reference has been made to a pair $\langle S_2, <_2 \rangle$, subsequent uses of \mathbf{S}_2 refer to this frame.

Valuations can be thought of as assigning truth values to atoms; this assignment needs to be extended to wffs of higher degree. The following definition is one of the standard ways of doing this. (Another — equivalent — definition of the extension that we will require occasionally is presented at the end of this section.)

Definition 2.2.3 *Given a language \mathcal{L} , a frame \mathbf{T} , and a valuation V for \mathcal{L} on \mathbf{T} , we extend V to a new function*

$$V^e : WFF_{\mathcal{L}} \times T \longrightarrow \{1, -1\} \text{ by:}$$

1. $V^e(\phi, t) = 1$ iff $t \in V(\phi)$, where $\phi \in ATOM_{\mathcal{L}}$
2. $V^e(\neg\phi, t) = -V^e(\phi, t)$
3. $V^e(\phi \wedge \psi, t) = \min\{V^e(\phi, t), V^e(\psi, t)\}$
4. $V^e(F\phi, t) = 1$ iff there is a t' such that $t < t'$ and $V^e(\phi, t') = 1$
5. $V^e(P\phi, t) = 1$ iff there is a t' such that $t' < t$ and $V^e(\phi, t') = 1$ □

In subsequent work we usually adopt the standard abuse of notation and talk of $V(\phi, t)$ when we really mean $V^e(\phi, t)$. We also treat $Val_{\mathcal{L}}(\mathbf{T})$ as though it contained the extended valuations, not the valuations simpliciter.

We now define our second fundamental semantic structure — models — and define the basic semantic notions concerning models and frames. Whereas frames are a naked temporal flow, models are a flow clothed in an information distribution.

Definition 2.2.4 (Models, truth and validity) *By a model \mathbf{M} for a language \mathcal{L} is meant an ordered pair $\langle \mathbf{T}, V \rangle$ where \mathbf{T} is a frame and V a valuation for \mathcal{L} on \mathbf{T} . \mathbf{T} is called the frame underlying the model. For $\phi \in WFF_{\mathcal{L}}$, we say that*

1. ϕ is true in a model \mathbf{M} at t iff $V^e(\phi, t) = 1$. Notation: $\mathbf{M} \models \phi[t]$.
2. ϕ is true in a model \mathbf{M} iff for all $t \in T$, $\mathbf{M} \models \phi[t]$. Notation: $\mathbf{M} \models \phi$.

3. ϕ is valid on a frame \mathbf{T} iff for all $V \in Val_{\mathcal{L}}(\mathbf{T})$, $\langle \mathbf{T}, V \rangle \models \phi$. Notation: $\mathbf{T} \models \phi$.
4. ϕ is valid on a class of frames \mathcal{T} iff for all $\mathbf{T} \in \mathcal{T}$, $\mathbf{T} \models \phi$. Notation: $\mathcal{T} \models \phi$.
5. ϕ is valid iff $\mathcal{U} \models \phi$ where \mathcal{U} is the class of all frames. Notation: $\models \phi$. □

All these definitions have obvious analogues for sets of wff Σ , and classes of models or frames; these will often be used in what follows. If $\mathbf{M} \models \phi[t]$ we sometimes say that \mathbf{M} *satisfies* ϕ , *verifies* ϕ , or *is a verifying model for* ϕ . We often talk of a model as having a certain relational structure, (such as being transitive), and talk of finite models, infinite models and so on. This means that the underlying frame has these properties. Similarly we often talk of the points of a model; these are simply the points of the underlying frame.

Definition 2.2.5 Let \mathbf{T} be a frame and \mathbf{M} be a model. Then:

$$\begin{aligned} Th_{\mathcal{L}}(\mathbf{T}) &= \{\phi \in WFF_{\mathcal{L}} : \mathbf{T} \models \phi\} \\ Th_{\mathcal{L}}(\mathbf{M}) &= \{\phi \in WFF_{\mathcal{L}} : \mathbf{M} \models \phi\} \end{aligned}$$

That is, the theory of a frame is the set of sentences valid on that frame; and the theory of a model is the set of sentences true in that model. □

If two frames (models) have identical theories we say they are *equivalent* frames (models). We extend the notion of theory to classes of frames (models) in the straightforward way. Let \mathcal{T} be a non-empty class of frames (models). Then:

$$Th_{\mathcal{L}}(\mathcal{T}) = \bigcap_{\mathbf{T} \in \mathcal{T}} Th_{\mathcal{L}}(\mathbf{T})$$

Both notions of Th take us from structures to syntactic entities. It is also useful to travel in the reverse direction, hence we define:

Definition 2.2.6 Let Σ be a set of sentences of some language. Then:

$$\begin{aligned} Fr(\Sigma) &= \{\mathbf{T} : \mathbf{T} \models \sigma, \text{ for all } \sigma \in \Sigma\} \\ Mod(\Sigma) &= \{\mathbf{M} : \mathbf{M} \models \sigma, \text{ for all } \sigma \in \Sigma\} \end{aligned}$$

If Σ is a singleton $\{\sigma\}$, we write $Fr(\sigma)$ and $Mod(\sigma)$ instead of $Fr(\{\sigma\})$ and $Mod(\{\sigma\})$.

□

The two most important consequence relations encountered in modal logic are *frame consequence* and *model consequence*. Frame consequence for a language \mathcal{L} of NTL is defined as follows:

$$\Sigma \models_f \phi \text{ iff } \forall \mathbf{T} \in \mathcal{U} (\forall \sigma \in \Sigma \ \mathbf{T} \models \sigma \Rightarrow \mathbf{T} \models \phi)$$

for all sets of \mathcal{L} -wffs Σ and \mathcal{L} -wffs ϕ .

Frame consequence is usually considered the more fundamental relation: it deals with *validity*, and hence abstracts completely away from the effects of any particular valuation. However since the work of Kit Fine and [27] S.K. Thomason [103] [104] in the early 1970s, it has been known that this consequence relation is not recursively enumerable for standard modal languages. This result is a fundamental one: Thomason later proved in [105] that modal logic is a reduction class of second-order logic and thus that ‘frame incompleteness’ does not reflect some vagary or weakness of known modal axiomatisations, but stems from the incompleteness of second order logic.

Now the non-axiomatisability of frame consequence for both purely Priorean and mixed languages follows immediately from these results. It will turn out, however, that frame consequence is a recursively enumerable relation for purely nominal languages: this is immediate by simple correspondence considerations. This is interesting, but as mixed languages are important, and as frame consequence is intractable for such languages, we will also need something like the weaker notion of model consequence:

$$\Sigma \models_m \phi \text{ iff } \forall \mathbf{M} (\forall \sigma \in \Sigma \ \mathbf{M} \models \sigma \Rightarrow \mathbf{M} \models \phi)$$

(Here \mathbf{M} is an arbitrary model.) In Chapter 4 we define the variant of this consequence relation that is needed.

To conclude this section we present the alternative method of extending valuations. The variant has an algebraic flavour. Note that in our previous definition V mapped atoms to subsets of some frame \mathbf{T} , but V^e did not do the same for arbitrary wffs —

rather, it invoked ‘truth values’, 1 and -1 .² There’s nothing wrong with this, but as the picture of the denotation of wffs being a set of times is rather attractive, let’s reformulate V^e along these lines. Obviously for atomic wffs we just let V^e agree with V . What sort of sets should complex wffs denote? The Boolean cases are clear: $V^e(\phi \wedge \psi)$ ought to be $V^e(\phi) \cap V^e(\psi)$, and $V^e(\neg\phi)$, should be $T \setminus V^e(\phi)$. The tense operators require a little more work.

Let $\mathbf{T} = \langle T, < \rangle$ be a frame and $S \subseteq T$. The *future projection* of S , $\pi_f(S)$, is defined by:

$$\pi_f(S) = \{t \in T : \exists s(s \in S \ \& \ t < s)\};$$

and the *past projection* of S , $\pi_p(S)$, is defined by:

$$\pi_p(S) = \{t \in T : \exists s(s \in S \ \& \ s < t)\}.$$

Formally, both π_f and π_p are operators on $Pow(T)$; they are an algebraicisation of the tense operators.³ Informally, if we regard S as a piece of information — or a ‘proposition’ if you will — the future projection of S is precisely the information that “It will be the case that S ”. Indeed, we can regard S as a wff of a propositional language, and treat $\pi_f(S)$ as a rather cumbersome notation for FS ; and indeed $\pi_f(S)$ is precisely the points of the frame where FS is true.⁴ Thus we define $V^e(F\phi) = \pi_f(V(\phi))$ and $V^e(P\phi) = \pi_p(V(\phi))$, and we have our new extended valuations. By and large we use

²So our earlier statement that V^e is an ‘extension’ of V was rather loose. The usage can be justified by its equivalence to the version of V^e about to be defined which really *is* an extension of V .

³There doesn’t seem to be a standard notation for these projections; this ‘ π ’ notation is from [5, page 141].

⁴This mutual translation between algebra and syntax is helpful in later work. It also proves useful to have at our disposal the operators dual to π_f and π_p , namely π_g and π_h defined by:

$$\pi_g(S) = \{t \in T : \forall s(t < s \Rightarrow s \in S)\}$$

the previous definition, but this new version will prove useful when we discuss ultrafilter extensions in the following chapter.

2.3 Some basic results

We now present four fundamental results that will be used frequently in what follows. The first three results are standard for intensional languages; the fourth, the Inheritance Lemma, is peculiar to sorted frameworks.

Lemma 2.3.1 (Agreement Lemma) *Let \mathcal{L} be a language of NTL, \mathbf{T} a frame, $\phi \in WFF_{\mathcal{L}}$ and $V, V' \in Val_{\mathcal{L}}(\mathbf{T})$ such that $V(a) = V'(a)$, for all atoms occurring in ϕ . Then:*

$$\langle \mathbf{T}, V \rangle \models \phi[t] \text{ iff } \langle \mathbf{T}, V' \rangle \models \phi[t]$$

for all $t \in T$.

Proof:

Induction on $deg(\phi)$. Only the base case of the induction that involves nominals is new, and this is immediate. \square

We next say what it is for two frames to be ‘structurally identical’.

and

$$\pi_h(S) = \{t \in T : \forall s (s < t \Rightarrow s \in S)\}.$$

Adopting the usual convention that for any $S \subseteq T$, $T \setminus S$ is written \overline{S} , we have that $\overline{\pi_f(S)} = \pi_g(S)$, $\overline{\pi_g(S)} = \pi_f(S)$, $\overline{\pi_p(S)} = \pi_h(S)$, and $\overline{\pi_h(S)} = \pi_p(S)$. This follows straight forwardly from the definitions of the projection operators, and shows that they really are dual pairs. In the transcription to tensed languages we see that these identities have as syntactical correlates such familiar facts as $F = \neg G \neg$, and so on.

Definition 2.3.1 (Isomorphisms) Let $\langle S, <_s \rangle$ and $\langle T, <_t \rangle$ be frames. A bijection f from S to T such that for all $s, s' \in S$:

$$s <_s s' \text{ iff } f(s) <_t f(s')$$

is said to be a frame isomorphism. If there exists a frame isomorphism between S and T we write $S \cong T$ and say the two frames are isomorphic.

Models $\langle S, V_s \rangle$ and $\langle T, V_t \rangle$ are said to be isomorphic iff there is a frame isomorphism f between S and T , and $V_t(a) = f[V_s(a)]$, for all $a \in \text{ATOM}_{\mathcal{L}}$. If two models M_1 and M_2 are isomorphic we write $M_1 \cong M_2$ and call f a model isomorphism. \square

A simple induction on $\text{deg}(\phi)$ establishes:

Lemma 2.3.2 $M_1 \cong M_2$ implies $\text{Th}(M_1) = \text{Th}(M_2)$. \square

As an immediate corollary we have that our languages cannot distinguish between isomorphic frames:

Corollary 2.3.1 $T_1 \cong T_2$ implies $\text{Th}(T_1) = \text{Th}(T_2)$.

Proof:

By the previous lemma if we can falsify a formula on one frame we can transfer the falsifying valuation to the other by means of the isomorphism. \square

As we shall see later, the converse result does not hold: equivalence does not imply isomorphism.⁵

⁵For the special case of finite *connected* frames, however, it does. (Connectedness is defined later in the chapter; essentially it means that starting at any point in the frame we can zig-zag our way to any other.) We refer the reader to [5, page 142] where the corresponding result is proved for Priorean languages. The addition of nominals does not affect the proof; in fact van Benthem's proof in effect proceeds by treating variables as (strong) nominals.

The next result gives each formula a ‘horizon’ — a limit past which it cannot see. The result will be useful to us on a number of occasions. We first need the following definition: ⁶

Definition 2.3.2 (n-hull) *Let $\mathbf{T} = \langle T, < \rangle$ be a frame and $t \in T$. $S_n(\mathbf{T}, t)$, the n-hull around t , is defined by:*

$$\begin{aligned} S_0(\mathbf{T}, t) &= \{t\} \\ S_{n+1}(\mathbf{T}, t) &= S_n(\mathbf{T}, t) \cup \{t' \in T : \exists t'' \in S_n(\mathbf{T}, t) \text{ s.t. } t' < t'' \text{ or } t'' < t'\} \end{aligned} \quad \square$$

The following result is obvious — and surprisingly tricky to prove cleanly. We leave its delights to the reader.

Lemma 2.3.3 (Horizon Lemma) *For any frame \mathbf{T} and any two valuations $V, V' \in \text{Val}(\mathbf{T})$ such that:*

$$V(a) \cap S_n(\mathbf{T}, t) = V'(a) \cap S_n(\mathbf{T}, t), \text{ for all atoms } a$$

then

$$\langle \mathbf{T}, V \rangle \models \phi[t] \text{ iff } \langle \mathbf{T}, V' \rangle \models \phi[t]$$

for all ϕ such that $td(\phi) \leq n$. \square

The next result, is very simple, and extremely important: it shows that ‘nominals inherit standard tense logic’. If a purely Priorean formula ϕ is valid on a frame \mathbf{T} , then any formula ϕ^n obtained from ϕ by uniformly substituting distinct nominals for all the variables in ϕ is valid on \mathbf{T} also. This should be clear: ϕ being valid on \mathbf{T} means that *for all* valuations V on \mathbf{T} , ϕ is true at all points. But in particular this means that for all valuations V such that V assigns *only singleton subsets of T to variables*, ϕ is true at all points — which is precisely what it means for any ϕ^n to be valid on \mathbf{T} . This result is worth pinning down a little more precisely:

⁶This definition, and the lemma that follows are temporal versions of those given for modal logic by van Benthem in [6, page 29].

Lemma 2.3.4 (Inheritance Lemma) *Let $\mathcal{L} = \langle \text{LOG}, \text{VAR}, \text{NOM} \rangle$ be a language of NTL such that $\text{VAR} \neq \emptyset$. Let \mathcal{L}^n , the nominalisation of \mathcal{L} , be the purely nominal language $\langle \text{LOG}, \emptyset, \text{VAR} \cup \text{NOM} \rangle$. For any wff ϕ of \mathcal{L} , let ϕ^n denote the same set theoretic entity in \mathcal{L}^n ; note that ϕ^n is a purely nominals wff of \mathcal{L}^n . Then for any frame \mathbf{T} we have that: $\mathbf{T} \models \phi$ implies $\mathbf{T} \models \phi^n$.*

Proof:

Firstly note that the claim in the statement of the above lemma that ϕ^n is a purely nominal wff of \mathcal{L}^n is trivial by induction on $\text{deg}(\phi)$; thus the statement of the lemma makes sense.

Secondly note that if $V \in \text{Val}_{\mathcal{L}^n}(\mathbf{T})$ then $V \in \text{Val}_{\mathcal{L}}(\mathbf{T})$; these valuations are the same set-theoretic entity. So the following statement makes sense:

$$\langle \mathbf{T}, V \rangle \models \phi[t] \text{ iff } \langle \mathbf{T}, V \rangle \models \phi^n[t]$$

for all purely Priorean \mathcal{L} wffs ϕ , frames \mathbf{T} , points $t \in T$, and valuations V on \mathbf{T} . Moreover, it is also a true statement, as a simple induction on $\text{deg}(\phi)$ shows.

Now our desired result is immediate: $\mathbf{T} \models \phi$ means that for all $V \in \text{Val}_{\mathcal{L}}(\mathbf{T})$ we have that $\langle \mathbf{T}, V \rangle \models \phi[t]$ for all $t \in T$. But $\text{Val}_{\mathcal{L}^n}(\mathbf{T}) \subseteq \text{Val}_{\mathcal{L}}(\mathbf{T})$, and by the equivalence just noted we are through. \square

Note that the proof of the Inheritance Lemma did not depend in any way on the fact that nominals are the ‘uniquely denoting sort’; it hinged solely on the fact that *nominals are a ‘subsort’ of variables*. All nominal valuations are variable valuations, and this was all that was needed. Thus the Inheritance Lemma is a special case of a wider lemma applicable to sorted intensional languages: schemas valid in a sort s remain valid when instantiated in any subsort s' of s . We will see another example of this shortly when we consider weak nominals.

The converse of the Inheritance Lemma does *not* hold — life would be very dull if it did. To take a rather boring example, the variable p is not valid on any frame at all, whereas its nominalisation i is valid on all and only the frames of cardinality 1. The

two sorts differ markedly in their properties, and indeed the next three chapters are essentially an exploration of these differences.

2.4 Filtrations and clusters

Filtrations are one of the most fundamental tools in intensional logic. Formulated originally by E.J. Lemmon and Dana Scott [61] for modal languages, they were generalised by Krister Segerberg in a series of papers [95] [96] [97] in which they were used to solve some difficult completeness and decidability problems in modal and tense logic. Their importance — and their ubiquity — has grown steadily since. They are useful in modal model theory, (see [6, pages 35–37] for some examples and a slightly more abstract view of filtrations), and have been adapted to many new types of intensional language. (For example, Segerberg has applied them to two dimensional modal logic [99]; and a discussion of filtrations for temporal logics of concurrency and Propositional Dynamic Logic may be found in [40].)

We now formulate the filtration method for languages of NTL. There are two motives for doing this. The first is practical: filtrations will prove as useful in NTL as they have elsewhere. For example, we will shortly use them to show that the set of NTL validities in any language is recursive. The second motive is that it provides a useful foretaste of the approach adopted in the next three chapters. A major concern of this thesis is to investigate which aspects of intensional logic are affected, and which unaffected, when nominals are added. To give a mixed bag of examples, we consider generated subframes, canonical Henkin frames, ultrafilter extensions, bulldozing, the Makinson construction and unravelling. The effects of the addition vary widely; sometimes there is no change, (generated subframes, ultrafilter extensions), sometimes the method fails, but can be suitably adapted (canonical Henkin models), and sometimes there is a breakdown (unravelling). Note, however, that all these widely differing effects are traceable to a single cause: the simple constraint we have placed on the interpretation of nominals. Time and again when faced with a new construction we will have to ask ourselves whether

the functions between language and frame they utilise are, in fact, valuations. If they are, the methods tend to transfer; if they aren't, we often face a major difficulty, namely that of altering the method so it does yield a valuation. Filtrations are a good starting point, as they are one of the pleasant cases: the work engendered by the addition of nominals is negligible, and basic filtration theory is unchanged.

Filtrations are a way of starting with a large model that verifies a set of wffs, and turning it into a smaller model — most usefully, a finite model — verifying that same set.

Definition 2.4.1 (Filtrations) Let \mathcal{L} be a language of NTL, $\mathbf{M} = \langle \mathbf{T}, V \rangle$ an \mathcal{L} -model, and Σ a set of \mathcal{L} -wffs closed under subformulas. Define an equivalence relation \sim on T by $t \sim t'$ iff $V(\phi, t) = V(\phi, t')$, for all $\phi \in \Sigma$ and $t, t' \in T$. Let $E(t)$ denote the equivalence class of t .⁷ Define $F = \{E(t) : t \in T\}$. Now suppose that $<_f$ is a binary relation on F satisfying:

1. $s < t \Rightarrow E(s) <_f E(t)$
2. $E(s) <_f E(t) \Rightarrow ((F\phi \in \Sigma \ \& \ \mathbf{M} \models \phi[t]) \Rightarrow \mathbf{M} \models F\phi[s])$
3. $E(s) <_f E(t) \Rightarrow ((P\phi \in \Sigma \ \& \ \mathbf{M} \models \phi[s]) \Rightarrow \mathbf{M} \models P\phi[t]).$

and further suppose that $V_f : \text{ATOM}_{\mathcal{L}} \rightarrow \text{Pow}(F)$ is a function satisfying:

1. $E(t) \in V_f(p)$ iff $t \in V(p)$, for all $p \in \text{VAR}_{\mathcal{L}}$
2. $E(t) \in V_f(i)$ iff $t \in V(i)$, for all $i \in \Sigma \cap \text{NOM}_{\mathcal{L}}$
3. $V_f(i)$ is a singleton subset of F , for all $i \in \text{NOM}_{\mathcal{L}} \setminus \Sigma$.

Then $\mathbf{M}^f = \langle \langle F, <_f \rangle, V_f \rangle$ is called a filtration of \mathbf{M} through Σ . □

The first thing we need to establish is that we have a supply of filtrations. Given a model \mathbf{M} and a subformulae closed set of sentences Σ , is a filtration of \mathbf{M} through Σ

⁷It would be more precise to subscript both E and \sim by Σ to indicate the relativisation of the definitions to that set of wffs; but this is far too cumbersome. Context, and explicit comment where necessary, will disambiguate.

always guaranteed to exist? Now, clearly, given any such pair we can form F and V_f ; the potential difficulty lies with $<_f$. Is it always possible to find a relation on F satisfying the constraints demanded above? The following lemma due to Segerberg [95, pages 11–12] answers the question positively by providing two relations that will always work. These relations, $<_\sigma$ and $<_\lambda$, give rise to the filtrations M^σ , and M^λ , called the *smallest* and the *largest* filtration of M through Σ respectively.⁸

Lemma 2.4.1 (Filtration existence) *Let \mathcal{L} be a language of NTL, $M = \langle \langle T, < \rangle, V \rangle$ an \mathcal{L} -model, and Σ a set of \mathcal{L} -wffs closed under subformulas. Form $F = \{E(t) : t \in T\}$ as in the filtration definition. Define two relations, $<_\sigma$ and $<_\lambda$ on F as follows:*

1. $E(s) <_\sigma E(t)$ iff $\exists s' \in E(s) \exists t' \in E(t) (s' < t')$
2. $E(s) <_\lambda E(t)$ iff $((F\phi \in \Sigma \ \& \ M \models \phi[t]) \Rightarrow M \models F\phi[s])$ and $((P\phi \in \Sigma \ \& \ M \models \phi[s]) \Rightarrow M \models P\phi[t])$.

Then, for any function $V_f : \text{ATOM}_{\mathcal{L}} \rightarrow \text{Pow}(F)$ satisfying the constraints on functions of this type given in the filtration definition, we have that both $M^\sigma = \langle \langle F, <_\sigma \rangle, V_f \rangle$ and $M^\lambda = \langle \langle F, <_\lambda \rangle, V_f \rangle$ are filtrations of M through Σ . \square

The reason for calling these filtrations ‘smallest’ and ‘largest’ is that given any other filtration M^f of M through Σ , $<_\sigma \subseteq <_f \subseteq <_\lambda$.⁹ Note that if \mathcal{L} is a language with nominals we cannot always, strictly speaking, talk of *the* largest filtration and *the* smallest filtration; for if $\text{NOM}_{\mathcal{L}} \setminus \Sigma$ is non-empty and F is not a singleton we may ‘freely assign’ what we like to the nominals, thus forming different filtrations. However all such variants share the same relational base, and we can talk of the smallest or the largest filtration ‘up to free assignment of nominals’ if we want to be very precise.

⁸This notation — σ for ‘smallest’ and λ for ‘largest’ — is taken from Goldblatt [40, page 32]. Note that the use of σ here as shorthand for ‘smallest’ has nothing to do with the set of sentences Σ through which we are filtrating.

⁹This is not difficult to prove; further details may be found in [95, pages 11–12]. Note that it can happen that $<_\sigma = <_\lambda$.

So, filtrations always exist — but are they models? We know from the above discussion that $\langle F, <_f \rangle$ is a well defined frame, and clearly V_f is a perfectly respectable function — but does V_f assign singletons to nominals?

Lemma 2.4.2 (Filtrations are models) *Let \mathcal{L} be a language of NTL, \mathbf{M} be an \mathcal{L} -model, Σ a set of \mathcal{L} -wffs closed under subformulas, and \mathbf{M}^f a filtration of \mathbf{M} through Σ . Then \mathbf{M}^f is a model.*

Proof:

We need merely check that V_f is a valuation. For any variables in the language its definition is clearly unproblematic, as variables may denote any subset of F . So suppose \mathcal{L} contains nominals. Suppose $i \in \Sigma$, and that $V(i) = \{t\}$. Clearly by the ‘if’ direction of the second clause for V_f there is at least one point of F in $V_f(i)$, namely $E(t)$. Equally clearly, by ‘only if’ direction of the same clause, there is no other; as otherwise we would have that $V(i)$ contained more than one element, and as V is a valuation this is impossible. Thus V_f handles all the nominals in Σ correctly; and by design V_f ‘freely assigns’ singletons to any nominals not in Σ ; hence V_f is a valuation, and \mathbf{M}^f a model.

□

This simple result is our admission ticket to the world of filtration theory. In particular, we can sprinkle references to languages of NTL through the statement of the following well known theorem from which the utility of filtrations ultimately stems:

Theorem 2.4.1 (Filtration theorem) *Let \mathcal{L} be a language of NTL, $\mathbf{M} = \langle \langle T, < \rangle, V \rangle$ a \mathcal{L} -model, Σ a set of \mathcal{L} -wffs closed under subformulas, and $\mathbf{M}^f = \langle \langle F, <_f \rangle, V_f \rangle$ a filtration of \mathbf{M} through Σ . Then:*

$$\mathbf{M} \models \sigma[t] \text{ iff } \mathbf{M}^f \models \sigma[E(t)],$$

for all $\sigma \in \Sigma$, and all $t \in T$.

Proof:

Induction on $\text{deg}(\sigma)$.

□

We can now show that there is an effective procedure for determining whether a wff γ of some language of NTL is valid. The result is a classic application of filtration theory; the argument is due to Lemmon and Scott [61].¹⁰ The result stems from two simple observations. The first is that if the (subformulae closed) set of sentences Σ through which we are filtrating is finite, the filtration is finite also. This shows that the set of wffs which *aren't* valid is recursively enumerable (re).¹¹ Now in the next chapter correspondence considerations will show us that the set of validities is r.e. as well; these two results taken together show that the validities form a recursive set. But in fact we don't need to appeal to correspondence theory to give us this result; a second observation about filtration arguments will take us the whole way. Given that the set Σ we are filtrating through is finite and of cardinality n , not only is any filtration through Σ finite, but it must have cardinality at most 2^n . (Briefly, this is because every point in a filtration consists of all the points in the original model that agree on all the information in Σ ; but given that there are n items of information in Σ , there are only 2^n possible ways of voting about them.) But this makes an important difference to the computability argument of the previous footnote: we now know that there is a bound on the search space. If γ has n subformulas, we need only search through all finite models up to size 2^n ; if γ is valid on all such models, it is valid tout court.

For many applications of filtrations it is important to be able to construct filtrations M' that inherit desirable relational properties of the original model M ; for example, it is often necessary to form a transitive filtration out of a transitive model. For some

¹⁰There is a good discussion, with generalisations, of this type of argument in [40, pages 30–35]. For further discussion see [107].

¹¹For suppose γ isn't valid. Then γ is falsifiable in some model M at some point t . Let Γ be the set of sentences consisting precisely of γ and all its subformulas; clearly Γ is finite, and thus any filtration of M through Γ is finite also. As filtrations always exist, we thus have by the Filtration Theorem that γ is falsified on a finite model at $E(t)$. It is easy to describe a program which systematically tests every wff on every finite model; this program thus enumerates the non-valid wffs.

important relational properties inheritance is automatic. Clearly if \mathbf{M} is a reflexive frame then any filtration \mathbf{M}' of \mathbf{M} (through any set Σ of sentences closed under subformulas) will be reflexive also; this follows from the first relational clause in the definition of filtrations. Equally clearly, again by the first clause, any filtration of a *trichotomous* model \mathbf{M} — that is, a model whose underlying frame ordering satisfies $\forall st(s < t \vee s = t \vee t < s)$ — is trichotomous also. In the case of transitivity, however, inheritance is not automatic. Nonetheless, if \mathbf{M} is a transitive model the relation $<_r$ on F (the set of equivalence classes induced by the set of sentences Σ under consideration) defined by:

$$E(s) <_r E(t) \quad \text{iff} \quad ((F\phi \in \Sigma \ \& \ \mathbf{M} \models \phi \vee F\phi[t]) \Rightarrow \mathbf{M} \models F\phi[s] \text{ and} \\ ((P\phi \in \Sigma \ \& \ \mathbf{M} \models \phi \vee P\phi[s]) \Rightarrow \mathbf{M} \models P\phi[t])$$

gives rise to a transitive filtration.¹² Following Segerberg [95, page 314] we call these Prior filtrations.

A concept that will prove vital in Chapter 5 is that of a *cluster*, a maximal equivalence relation on a frame.

Definition 2.4.2 (Clusters) *Let $\langle T, < \rangle$ be a frame. Then any $C \subseteq T$ is called a cluster iff $< \cap C^2$ is an equivalence relation, and this is not true for any proper superset C' of C . A cluster is proper iff it contains at least two elements, and is simple iff it consists of a single reflexive point.* \square

Concrete examples will make it clear why this is a useful concept; for the time being note that if we filtrate an infinite model through a finite set of sentences we have formed a finite model out of an infinite one. Thus a lot of identifications have taken place and we have probably formed clusters.

¹²The proof is a straightforward modification of the proof of filtration existence, using the fact that $FF\phi \rightarrow F\phi$ is valid on all transitive frames.

2.5 Unraveling and paths

In this section we discuss a method of model transformation that fails to transfer to languages with nominals; the method of unraveling. Among other things, unraveling is a means of turning an ordinary tense or modal logical model into an equivalent intransitive model. The frame transformation that underpins the method is best introduced by example, so I'll first display a simple modal unraveling of a frame.

Let 2_G be the frame $\langle \{0, 1, 2\}, \{\langle 2, 0 \rangle, \langle 2, 1 \rangle, \langle 1, 0 \rangle\} \rangle$. Now define $a = \langle 2 \rangle$, $b = \langle 2, 0 \rangle$, $c = \langle 2, 1 \rangle$, $d = \langle 2, 1, 0 \rangle$. That is, $\{a, b, c, d\}$ is the set consisting of all possible trips through the frame 2_G that start at 2. We define an ordering $<_u$ on this set by stipulating that given any two sequences s and s' , $s <_u s'$ iff $\text{len}(s') = \text{len}(s) + 1$, and s is an initial segment of s' . That is, s' must be a 'next step' continuation of the trip s . Thus we have that $a <_u b$, $a <_u c$ and $c <_u d$. We define 2_U to be the frame $\langle \{a, b, c, d\}, <_u \rangle$ and call 2_U the modal unraveling of 2_G about 2. As this example makes clear, unraveling is essentially a method of transforming an arbitrary frame into a tree.

Defining the temporal equivalent of unravelling is more complicated. This is because all the moves, both forwards and backwards in time, possible in the original frame must be encoded in the unraveling — tensed languages have backward looking operators, so this is essential. The definition now given, which uses 'arrow insertion' to code this bidirectionality, is van Benthem's [5, page 179]. In what follows \rightarrow and \leftarrow are just two convenient new symbols, and \rightarrow has nothing to do with material implication.

Let $\mathbf{T} = \langle T, <_t \rangle$ be a frame and $t_0 \in T$. Define a new frame $\mathbf{U} = \langle U, <_u \rangle$ where U is the smallest set such that $\langle t_0 \rangle \in U$ and:

$$\begin{aligned} \langle t_1, \dots, t_k \rangle \in U \text{ and } t_k <_t t_{k+1} &\text{ implies } \langle t_1, \dots, t_k, \rightarrow, t_{k+1} \rangle \in U \\ \langle t_1, \dots, t_k \rangle \in U \text{ and } t_{k+1} <_t t_k &\text{ implies } \langle t_1, \dots, t_k, \leftarrow, t_{k+1} \rangle \in U \end{aligned}$$

and $<_u$ consists of all and only the pairs of the form

$$\langle \langle t_1, \dots, t_k \rangle, \langle t_1, \dots, t_k, \rightarrow, t_{k+1} \rangle \rangle \text{ or } \langle \langle t_1, \dots, t_k, \leftarrow, t_{k+1} \rangle, \langle t_1, \dots, t_k \rangle \rangle.$$

We call U the *unraveling of T about t_0* . Note that a necessary condition for two elements in the unraveled frame to be related is that they differ in length by 2.

The importance of U is that no matter what frame T we unraveling, $<_u$ is always intransitive. For suppose $u_1 <_u u_2$ and $u_2 <_u u_3$. Inspection of the definition of $<_u$ shows that either u_1 and u_3 have the same length, or they differ in length by 4. Either way, $u_1 \not<_u u_3$, and U is intransitive.

Suppose $\langle T, V_t \rangle$ is a model for a purely Priorean language. Define a function $f : U \rightarrow T$ by setting $f(u)$ to be the last element in u , for all $u \in U$. Define a Priorean valuation on U by setting $u \in V_u(p)$ iff $f(u) \in V_t(p)$, for all variables p . We now have that for all wffs ϕ :

$$\langle U, V_u \rangle \models \phi[u] \text{ iff } \langle T, V_t \rangle \models \phi[f(u)],$$

as can be shown by induction on $\text{deg}(\phi)$.¹³

This equivalence, taken together with the fact that U is intransitive, shows that there is no purely Priorean wff valid on precisely the intransitive frames. In fact we can show that a wff ϕ is valid on the class of intransitive frames iff it is (universally) valid. The right to left implication is immediate. To show the other direction, assume that ϕ is not valid. This means that in some model $\langle T, V_t \rangle$ at some point $t_0 \in T$, $\langle T, V_t \rangle \not\models \phi[t_0]$. But by unravelling $\langle T, V_t \rangle$ through t_0 we form an intransitive model that refutes ϕ , and hence ϕ is not valid on the intransitive frames. Thus, if ϕ is valid on all the intransitive frames, it is also valid.

This result does not hold for languages with nominals. It is easy to check that $T \models FF_i \rightarrow \neg Fi$ iff T is intransitive. Languages with nominals have a grip on intransitive frames not possible in Priorean languages. More generally, languages with nominals ‘see’ frames very differently from purely Priorean languages, and this vision does not survive the unraveling process.

¹³This equivalence also follows from more general considerations. In the terminology to be introduced in the next chapter, f is a p-morphism from the model $\langle U, V_u \rangle$ to the submodel of $\langle T, V_t \rangle$ generated by t_0 , which guarantees equivalence.

There's an obvious reason why unraveling fails: the way unraveled *models* were defined will not in general extend to languages with nominals. Suppose the function f described above is not injective. Then augmenting the above definition of V_u by stipulating that $u \in V_u(i)$ iff $f(u) \in V_i(i)$, for all nominals i does not yield an NTL valuation — V_u does not assign singletons to nominals and so we haven't succeeded in building a model on the unraveled frame.

More interestingly, considering unraveling leads naturally to an intuition that will play a motivational role throughout the thesis: the importance of *paths* for languages with nominals.

Definition 2.5.1 (Paths) *By a path through a frame $\langle T, < \rangle$ is meant any finite sequence of points of T such that for every pair t_m, t_{m+1} in the sequence, either $t_m < t_{m+1}$ or $t_{m+1} < t_m$. The length of a path is the sequence length. A frame is connected iff there exists a path between any two of its points.* \square

That is, a path through a frame is finite sequence of moves forwards and backwards in time. We often emphasise the bidirectionality of the concept in what follows by calling paths *zig-zag paths*.

When we consider the minimal logic for languages of NTL in Chapter 4 we shall see that the axiomatisation needed is the ordinary minimal tense logical axiomatisation augmented by axioms that can be regarded as *path equations* — axioms asserting that certain paths must be coterminous. More generally, we shall see that languages with nominals are very good at expressing the existence or non-existence of certain paths — the intransitivity defining wff $FFi \rightarrow \neg Fi$ is a good example of this. Given the importance of paths for languages with NTL it is hardly surprising that unraveling fails: unraveling is a method that systematically destroys path equations.

2.6 Variants and extensions

Intuitively, we make a language of *nominal modal logic* (NML) by taking an ordinary modal language and adjoining a set of new atoms called nominals. More formally, we define a language \mathcal{L} of NML to be a triple $\langle LOG, VAR, NOM \rangle$ satisfying the conditions given in the definition of languages of NTL, with the sole difference that LOG is now a set of cardinality 5. We represent the elements of LOG by $\wedge, \neg, \Diamond, \text{, and } ($. Once again, this definition admits two extreme cases; $NOM = \emptyset$, and $VAR = \emptyset$, which give rise to languages we call ‘purely Kripkean’ and ‘purely nominal modal’ respectively. The needed syntactic concepts — such as $WFF_{\mathcal{L}}$, degree, and subformula — are straightforwardly adapted from those given for NTL; essentially, we identify \Diamond with F and forget all references to P .

The semantics of languages of NML is given in terms of frames and models; again we insist that for a function V from atoms of \mathcal{L} to the powerset of some frame \mathbf{T} to be a valuation, that it must assign singletons of T to any \mathcal{L} -nominals. With this stipulated, one can adapt all the definitions of semantic concepts — such as truth at a point, and validity — from those given for NTL by simply ignoring all reference to the ‘backward looking’ operator P , and reading F as \Diamond . In particular, with this change made the definition of filtrations for languages of NML is correct, as are the results concerning them.¹⁴ By a *modal path* through a frame $\langle \mathbf{T}, < \rangle$ is meant a finite sequence of points of T such that for all pairs of points t_k, t_{k+1} in the sequence, $t_k < t_{k+1}$. In contrast to temporal paths, modal paths are unidirectional.

In what follows we usually point out the corresponding concepts and results for languages of NML of the temporal work that is our primary concern. By and large the adaptations are routine — just as we have noted in the case of filtrations — and consist of ignoring clauses concerning P . In some cases, however, the differences between the

¹⁴The modal analogue of Prior filtrations are usually called Lemmon filtrations.

two frameworks are rather more interesting. For example, simpler axiomatisations are possible for minimal NTL than for minimal NML, as we prove in Chapter 4.

By a *multimodal language* is meant a language of propositional calculus augmented by indexed one place sentence operators. If the *index set* of the language is Δ — that is, the one place sentence operators available are $\{\Diamond_\delta : \delta \in \Delta\}$ — then we extend the formation rules of propositional calculus by adding the clause: ‘If ϕ is a wff, then so are $\Diamond_\delta\phi$, for all $\delta \in \Delta$ ’. In short, multimodal languages look like modal languages, save that there are lots of dangling indices — a pattern repeated in their semantics.

The basic semantic entity is the *multiframe*, a set bearing multiple binary relations — that is, instead of one accessibility or ordering relation on points, multiframe have many. A good example of a multiframe is the map of the London Underground. Here, the nodes representing the tube stations are the points, and the different lines — the Victoria line, the Central line, the Bakerloo line and so on, (all conveniently in different colours) — are the different accessibility relations. More precisely, if \mathcal{L} is a multimodal language with index set Δ , then by a Δ -multiframe is meant a pair $\langle T, \{\prec_\delta\}_{\delta \in \Delta} \rangle$ where T is a non-empty set, and for each $\delta \in \Delta$, \prec_δ is a binary relation on T . Valuations, as in ordinary modal logic, are functions that assign arbitrary subsets of frames to propositional variables. The truth definition is the expected one, the key clause being:

$$V(\Diamond_\delta\phi, t) = 1 \text{ iff } \exists t'(t \prec_\delta t' \ \& \ V(\phi, t') = 1).$$

For further details of multimodal languages the reader is referred to [40, Chapter 5] and [97].

Multimodal frameworks arise naturally in a number of applications; probably the earliest example is Hintikka’s [45] epistemic logic. Here operators K_α are introduced with the reading ‘Agent α knows that ...’. Thus Δ is the set of epistemic agents, (man, beast or machine); and the \prec_δ relation links a state t to an agent α ’s epistemic alternatives at t . Tense logic is frequently viewed as a multimodal logic. Here we suppose that our index set is $\{p, f\}$; thus we have two operators \Diamond_p and \Diamond_f , and frames have the form $\langle T, \{\prec_p, \prec_f\} \rangle$. To make this look like a flow of time we insist that the only

admissible frames are those where $<_p$ and $<_f$ are converse relations.¹⁵ Finally, and most importantly, Propositional Dynamic Logic (PDL) beautifully exploits multiframe. The details would take us too far afield however, and we refer the reader to [78], [100], or [40, Chapter 10].

Extending the multimodal languages to multimodal languages with nominals is trivial — we merely add nominals and interpret them as singletons; we take the definition as read. We examine two multimodal extensions of NTL later in the thesis.

We now consider two natural variants of NTL that will crop up from time to time in work that follows: weak languages of NTL, and strong languages of NTL.

Weak languages of NTL are syntactically identical to ordinary languages of NTL. That is, a triple $\mathcal{L} = \langle LOG, VAR, NOM \rangle$ may be regarded as — *is*, syntactically — both a weak language, and an ordinary language of NTL; and further, $WFF_{\mathcal{L}}$ is both the set of weak wffs, and ordinary wffs. We export our usual syntactic definitions unchanged to weak languages. Some obvious variants of our usual terminology will be used without further comment — for example we talk of weak nominals.

Weak languages differ from ordinary NTL, in the way they are interpreted; in particular in weak languages nominals are true at *at most one* point, not exactly one. If you like, weak nominals are names, but names with a built in possibility of referential failure. More precisely:

¹⁵I don't think that this is a particularly good way of thinking about tense logic. For example, the multimodal conception of tense logic leads fairly directly to the usual axiomatisation of minimal tense logic, presented in Chapter 4. One can do a lot better than this, as recent work by Humberstone shows [49], by exploiting the fact that tensed languages can look in both directions. If anything, the notion of a 'multitensed' language seems a better generalisation than that of a multimodal language. Such a conception would view multimodal languages as the forward looking fragments of multitensed languages.

Definition 2.6.1 (Weak Valuations) *Let \mathcal{L} be a weak language and \mathbf{T} a frame. By a weak valuation V for \mathcal{L} on \mathbf{T} is meant a function*

$$V : \text{ATOM}_{\mathcal{L}} \longrightarrow \text{Pow}(\mathbf{T})$$

such that for all $i \in \text{NOM}_{\mathcal{L}}$, $V(i)$ is either a singleton subset of \mathbf{T} or \emptyset . \square

The relationship between weak and ordinary languages of NTL is straightforward. Firstly, they are identical as regards validity:

Lemma 2.6.1 *Let \mathcal{L} a triple $\langle \text{LOG}, \text{VAR}, \text{NOM} \rangle$ of the usual type. Writing $\models_w \phi$ to indicate that ϕ is weakly valid we have that $\models_w \phi$ iff $\models \phi$.*

Proof:

To see that all weak validities are validities, simply note that all ordinary NTL valuations on an arbitrary frame \mathbf{T} are weak valuations; hence, (as with our proof of the Inheritance Lemma), weak validity implies validity.

The converse, as might be expected from the fact that not all ordinary NTL valuations are weak valuations, is more difficult. In fact it is a corollary of the completeness proof for minimal NTL we give in Chapter 4. \square

Nonetheless, the two types of language are not expressively equal, as we shall see in the next chapter. In particular, in languages with ordinary nominals we can write down a formula valid on precisely the trichotomous frames; whereas this is not possible in any weak language.

In short, with weak nominals we have encountered another sort: ordinary nominals are a subsort of this new sort, and are more expressive. With strong languages, matters are more complex.

Strong languages of NTL have already been mentioned; they are languages where not only do nominals name points of the frame, but *every point of the frame is named by some nominal*. Formally, we define a language \mathcal{L} of strong NTL to be a triple

$\langle LOG, VAR, NOM \rangle$ of the type used to define languages of ordinary NTL, save that we demand that $NOM \neq \emptyset$. (Trivially we cannot strongly interpret a language without nominals on any frame.) We form $WFF_{\mathcal{L}}$ as before and use the usual NTL syntactic definitions unchanged.

The key semantic definition is that of a *strong valuation*:

Definition 2.6.2 (Strong Valuations) *Let \mathcal{L} be a language of strong NTL, \mathbf{T} a frame, and V a valuation of \mathcal{L} on \mathbf{T} . We say that V is a strong valuation of \mathcal{L} on \mathbf{T} iff $\bigcup_{i \in NOM_{\mathcal{L}}} V(i) = T$; and in such a case we say that V covers \mathbf{T} .*

Clearly we cannot strongly interpret an arbitrary language on an arbitrary frame; in order for a strong valuation of \mathcal{L} on \mathbf{T} to exist we must have that $card(T) \leq card(NOM_{\mathcal{L}})$. When (and only when) this condition obtains we say that \mathcal{L} covers \mathbf{T} . The usual semantic definitions go through virtually unchanged, with references to ‘valuations’ being replaced by references to ‘strong valuations’. Only in defining the notion of validity do we need a little care; here we need to say that a wff ϕ of some strong language \mathcal{L} is valid iff ϕ is true at all points t on all frames \mathbf{T} such that \mathcal{L} covers \mathbf{T} , for all strong valuations V .

Are there more strong validities than validities, and are strong languages more expressive than ordinary languages of NTL? The answer depends on the number of nominals that \mathcal{L} contains. If \mathcal{L} is what is traditionally regarded as a ‘reasonable language’, that is, a language in which we have at our disposal as many atomic symbols (in this case nominals) as we require, the answer is ‘no’. Strong languages aren’t more expressive than ordinary languages and give rise to no new validities:

Lemma 2.6.2 *Let \mathcal{L} be a strong language with a countably infinite collection of nominals and \mathbf{T} a frame that is coverable by \mathcal{L} . Writing $\mathbf{T} \models_s \phi$ to indicate that ϕ is strongly valid on \mathbf{T} , we have that $\mathbf{T} \models_s \phi$ iff $\mathbf{T} \models \phi$.*

Proof:

That $\mathbf{T} \models \phi$ implies $\mathbf{T} \models_s \phi$ is clear by an argument analogous to those used in proving the Inheritance Lemma, and that weak validity implies validity, namely: as all strong valuations of \mathcal{L} on \mathbf{T} are ordinary NTL valuations of \mathcal{L} on \mathbf{T} , if ϕ is ordinary NTL valid on \mathbf{T} it must be strongly valid on \mathbf{T} as well.

To see the converse, argue by contrapositive. Suppose $\mathbf{T} \not\models \phi$, that is, for some valuation V on \mathbf{T} and some $t \in T$, $\langle \mathbf{T}, V \rangle \not\models \phi[t]$. Now there are a countable infinity of \mathcal{L} -nominals that *don't* occur in ϕ , and \mathbf{T} contains at most a countable infinity of points, so we can find a valuation V' that agrees with V on all atoms occurring in ϕ but that covers \mathbf{T} . By the Agreement Lemma, $\langle \mathbf{T}, V' \rangle \not\models \phi[t]$, and V' is a strong valuation. \square

However there's clearly a potential problem for languages with a *finite* number of nominals in constructing an analog of the second part of the previous proof. How could we form the strong valuation V' if ϕ contained all the \mathcal{L} nominals? And, indeed, there is a problem here. For example, if \mathcal{L} is a language of strong NTL with exactly two nominals, then $\neg(i \wedge j) \wedge (i \vee j)$ is valid on precisely the frames of cardinality 2. In the following chapter we will see in *no* language of ordinary NTL, not just the finite ones, is there a formula valid on precisely the frames of cardinality two; thus finite languages of strong NTL are more expressive^{than} their ordinary counterparts — at least as regards coverable frames.

The preceding observation isn't particularly important in itself — we aren't very interested in languages of finite cardinality, nor in finite frames — but it does help to indicate that there is something rather odd about the strong interpretation. Two phrases should make the source of the strangeness plain: in strong languages we have effectively insisted that *our language is larger than our world*, and secondly, we have placed a *global* restriction on the interpretation of certain atoms that is really quite unusual.

Clearly 'language is bigger than world' for strong languages, and this is rather strange — it certainly accounts for the odd results for languages with a finite number of strong nominals. Whether such a state of affairs is undesirable is another matter. Certainly as far as natural language semantics is concerned it seems peculiar; there we are used to thinking of human languages as small, and the universe as big. However for other

applications, such as reasoning about program behaviour, it may be quite natural. In such applications the universe of discourse is itself a construct, and inherently ‘small’; and the boundary between constructs, and representations of constructs, not so clearly drawn — or at least, traditional notions of where to draw the boundary may be irrelevant or unhelpful. In such cases it is probably natural to insist on a far tighter link between language and model than is traditional.

More than mere sorting is involved in strong valuations. Certainly there is sorting in such languages — strong nominals obey their usual constraint — but in addition there is the global covering demand. This demand is rather odd, if only because the individual nominals clearly don’t interact syntactically amongst themselves in any interesting way. By way of comparison, consider the first order language of arithmetic, interpreted on the standard model. Now here we have names for every element: 0 , $s(0)$, $s(s(0))$, ... and so on; the standard model is ‘covered’. But there is a structure on these terms, a structure that mirrors the additive structure of the natural numbers. The connection between terms is part of the logic: $s(0) + s(s(0))$ really is $s(s(s(0)))$. All this is missing in the case of strong nominals; they must cover their frame, but they do so blindly. Nothing in the syntax reflects this, and no structure of the frame is being seen save cardinality. Nonetheless the idea of global constraints — as opposed to the local sortal constraints — is an interesting one, though it is not pursued in this thesis. We won’t discuss strong languages any more, though in Chapter 5 we shall see that strong valuations can be highly desirable for technical reasons.

Apart from being used occasionally in Chapter 7 to represent natural language sentences, first order languages of NTL are hardly considered in this thesis. I briefly introduce them now largely in order to draw a distinction between referential sorts and information bearing sorts.

By a language of first order NTL is meant a purely nominal language to which some first order language has been adjoined. The wffs are made in the obvious way, and we could summarise what has happened by saying that the resultant language is like a mixed language of NTL save that all the variables have been replaced by first order formulas.

As to their semantics, for the sake of simplicity I'll assume that a constant domain of quantification is being used and that constants are rigid designators. The advantages of this assumption is that it leads to very simple logics: in particular, we can axiomatise the first order extensions of the logics described in Chapters 4 and 5 simply by adjoining the $Q1$ axioms and rules as described in James Garson's survey of first order modal logic [36, page 255 – 256]. The disadvantages of this semantics is that it is (allegedly) unintuitive as it forces us to admit the Barcan formula, $\forall x G\phi \rightarrow G\forall x\phi$. This may be so, but the alternatives, which often make use of free logic, can hardly be described as simple: the reader is referred to Garson's article for a thorough discussion of the options.

The interesting point is that we only replaced the variables by first order expressions. That is, we tacitly accepted that variables were an 'information bearing sort' whose ultimate fate was to be replaced by the wffs of a more expressive language, whereas nominals conveyed a different type of information — purely referential information — and should not so be replaced. This is an important distinction. When we discuss natural language semantics in Chapter 7, the only use we will make of sorting will be to use such propositional referential sorts to model some aspects of tenses and their interaction with temporally referring expressions. Nonetheless, there is no reason whatsoever why we should think of variables as a undifferentiated collection of placeholders, or that sorting is only useful to achieve reference. Indeed I argue in later chapters that we should attempt to sub-sort our variables to mirror the distinctions natural language draws between event types, and that imposing plausible constraints on information distributions (valuations) is just as important a part of temporal modeling as choosing a reasonable class of frames.

This concludes our preparatory work. We now are ready to investigate why sorted languages in general — and languages with nominals in particular — are logically interesting.

Chapter 3

Model Theory

The main aim of this chapter is to discuss the gains in expressive power that result when nominals are added to tensed languages. In the first section we display many examples of this new expressive power; in the second section we discuss validity and truth preserving transformations; and in the third section we examine correspondences between languages of NTL and other temporal languages. We conclude the chapter by discussing a possible direction for further work.

3.1 Definability in NTL

Frames are our most basic semantic entities, and a fundamental question to ask is what constraints our languages can impose on them. How precisely can our languages pin down the classes of frames we find interesting, and can we pick out more interesting classes of frames using nominals than we can without? The technical concept underlying these questions is *definability*:

Definition 3.1.1 (Definability) *Let \mathcal{T} be a class of frames. Let ϕ be a formula of a language of NTL (or NML). We say ϕ defines \mathcal{T} iff: $\mathbf{T} \models \phi$ iff $\mathbf{T} \in \mathcal{T}$. \square*

Some standard examples, together with their traditional names, will be useful. The class of transitive frames is defined by $FFp \rightarrow Fp$ (4); the class of reflexive frames by

$p \rightarrow Fp$ (T); and the class of symmetric frames by $FGp \rightarrow p$ (B). The class of Church-Rosser frames is defined by $FGp \rightarrow GFp$ (.2); the class of dense frames by $Fp \rightarrow FFp$ (A); and the class of frames such that every point has a successor — the right unbounded frames — by $F\top$ (D_r). The class of right unbounded frames is mirrored by the class of left unbounded frames, consisting of those frames in which every point has a predecessor. This is defined by $P\top$ (D_l).

A simple lemma allows us to create new definable classes out of old:

Lemma 3.1.1 (Intersective Closure Lemma) *Let ϕ and ψ be wffs of some language of NTL, and \mathcal{T} and \mathcal{T}' classes of frames such that ϕ defines \mathcal{T} and ψ defines \mathcal{T}' . Then $\phi \wedge \psi$ defines $\mathcal{T} \cap \mathcal{T}'$* □

The lemma follows simply from the definition of validity. As an example of its use, the class of all unbounded frames — those frames in which every point has both a successor and a predecessor — is defined by $P\top \wedge F\top$ (D).

It will prove useful to have at our disposal a certain first-order language, L_0 . This language has only a single non-logical symbol — a binary predicate $<$. We assume the equality predicate $=$ as a logical symbol.¹ Note that any frame $\mathbf{T} = \langle T, <_t \rangle$ is a first order structure for L_0 — T is the domain of quantification and $<_t$ the extension of the $<$ predicate — thus we can talk of a formula of L_0 being valid on a frame. Here validity means ordinary first order validity. We say that a class of frames \mathcal{T} is an L_0 expressible class iff for some wff ψ in L_0 , ψ is valid on precisely the frames in \mathcal{T} . For example, the class of right unbounded frames is an L_0 expressible class, for this is precisely the class of frames on which $\forall x \exists y (x < y)$ is valid. We often say that a wff of L_0 expresses a first order *condition on frames*, or even, is a (first order) condition on frames; and when we say that a sentence ϕ of NTL (or NML) defines a certain condition ψ , where ψ is a wff

¹There is scope for confusion here: $=$ is already being used for set theoretic equality in our semi-formal metalanguage; and in this language $<$ is standardly used for the relation on frames. We take care to ensure that context will disambiguate.

of L_0 , we simply mean that the NTL wff and the L_0 wff are valid on exactly the same frames. For example, $F\top$ defines the condition $\forall x\exists y(x < y)$.

It is important to note that not all classes of frames definable in standard tensed languages are L_0 expressible. For example, the class of frames which have a transitive and well-founded ordering relation can be defined by Löb's axiom, $H(Hp \rightarrow p) \rightarrow Hp$, and the usual appeal to the compactness theorem shows that no wff of L_0 expresses this condition. Ordinary tense logic is intrinsically higher order, and corresponds to a fragment of a certain second order language L_2 . On the other hand, as we will see later in the chapter, purely nominal sentences can define only L_0 expressible classes of frames.

3.1.1 Six Simple Conditions

Consider the following six conditions on frames: ²

Irreflexivity	$\forall x(x \not< x)$
Asymmetry	$\forall xy(x < y \rightarrow y \not< x)$
Antisymmetry	$\forall xy(x < y \wedge y < x \rightarrow x = y)$
Trichotomy	$\forall xy(x < y \vee x = y \vee y < x)$
Right Directedness	$\forall xy\exists z(x < z \wedge y < z)$
Right Discreteness	$\forall xy(x < y \rightarrow \exists z(x < z \wedge \neg\exists w(x < w < z)))$

None of these conditions is definable in a purely Priorean language; all of these conditions are definable in any language with nominals, as follows:

²The definitions are those of [5]; the single change is that the condition here called trichotomy is there called linearity. Corresponding to right directedness and discreteness are left directedness and discreteness, defined in the obvious way. By directedness (discreteness) is meant the condition expressed by the conjunction of left and right directedness (discreteness).

$i \rightarrow \neg Fi$	(I)
$i \rightarrow \neg FFi$	(Ass)
$i \rightarrow G(Fi \rightarrow i)$	(Anti)
$Pi \vee i \vee Fi$	(Tr)
FPi	(RDir)
$i \rightarrow (F\top \rightarrow FHH\neg i)$	(RDisc)

Left directedness and discreteness are defined by the mirror images of RDir and RDisc respectively. By Dir is meant the conjunction of RDir with its mirror image, and by Disc the conjunction of RDisc with its mirror image; by the Intersective Closure Lemma, these define the classes of directed and discrete frames respectively. The method of defining right discreteness given above is due to Inge Bethke.³

All six conditions are potentially useful constraints to impose when modeling temporal precedence. Some seem almost indispensable: for example it is arguable that the strictly *partially* ordered frames (SPOs) embody our minimal assumptions about temporal flow — in the ‘time as river’ metaphor, transitivity ensures the river flows, and irreflexivity ensures there are no whirlpools; the present, or any other time, does not precede itself. The gain of irreflexivity is an important one.

Equally interesting is the gain of discreteness. The choice between dense and discrete time is a fundamental branch point in temporal modeling. For many applications density, or even continuity, is a natural choice; but for others — such as modeling the execution of a computer program, or ‘calendar semantics’ for natural language — discrete time is appropriate. Indeed for some applications one might want both: for natural language semantics we might want a discrete calendar structure sitting on top of a dense time flow. The important point is that it’s useful to have both options definable in our languages, and with the aid of nominals both are.

³Previously I used $i \rightarrow (\neg Fi \wedge (F\neg i \rightarrow F(\neg i \wedge HH\neg i)))$, which works, but is clumsy.

Note that we can also define the partial orders (POs), total orders (TOs) or strict total orders (STOs) using nominals; for example, the class of STOs is definable by $I \wedge 4 \wedge \text{Tr}$. None of these classes are definable in purely Priorean languages.

The proofs that the NTL wffs given define the stated condition are straightforward. We conclude this section with an example.

Lemma 3.1.2 *T is right discrete iff $T \models i \rightarrow (F\top \rightarrow FHH\neg i)$*

Proof:

To show that **T** is right discrete implies $T \models \text{RDisc}$, assume that $V(i, t) = 1$ and $V(F\top, t) = 1$. Then there exists a $t' > t$ such that $V(\top, t') = 1$. Now as t has a successor t' it has an immediate successor s . But $V(HH\neg i, s) = 1$. For suppose $V(HH\neg i, s) = -1$, that is, $V(PPi, s) = 1$. Then there is a u such that $u < s$ and $V(Pi, u) = 1$. But this means that $t < u$ as i is true uniquely at t . Hence $t < u < s$ contradicting the choice of s as an immediate successor of t . So $V(HH\neg i, s) = 1$ and thus $V(FHH\neg i, t) = 1$, verifying RDisc.

To show the converse, suppose $T \models i \rightarrow (F\top \rightarrow FHHi)$. Let V be an arbitrary valuation on **T** such that $V(i) = \{t\}$. If t has no successors there is nothing to prove, so suppose there is a t' such that $t < t'$. Thus we have $V(i, t) = 1$ and $V(F\top, t) = 1$ and hence, as RDisc is valid on **T**, $V(FHH\neg i, t) = 1$. Hence there is a point $s > t$ such that $V(HH\neg i, s) = 1$. Let u be an arbitrary point in T . If $t < u < s$ then $V(H\neg i, u) = -1$ and thus $V(HH\neg i, s) = -1$; contradiction. Hence s is an immediate successor of t and thus **T** is right discrete. \square

3.1.2 Special Structures and counting

In this section we examine the class of strict total orders and some of its interesting subclasses, and consider some of the things we can count with nominals. First note that the class of STOs is definable by a purely nominal sentence. A quick check reveals that

$FFi \rightarrow Fi$ defines transitivity;⁴ so we define ϕ^T to be $(FFi \rightarrow Fi) \wedge I \wedge \text{Tr}$. By the Intersective Closure Lemma this purely nominal sentence defines the class of STOs. We now show that some important subclasses of the STOs can be defined.

Let ϕ^{T^*} be the sentence: $\phi^T \wedge (F\top) \wedge (P\top)$. This defines the class of unbounded STOs. Here we regard \top as being $i \vee \neg i$, hence this class of frames is definable by a purely nominal sentence. Now define $\phi^{T^{*d}}$ to be $\phi^{T^*} \wedge \text{Disc}$; clearly this purely nominal sentence defines the class of unbounded discrete STOs.⁵

Now \mathbf{Z} is in the class of unbounded discrete total orders, but so are other (non-isomorphic) structures — for example, the frame consisting of two copies of Z lying end to end with the obvious ordering. It is possible, *using a mixed sentence*, to eliminate such ‘pathologies’ and define \mathbf{Z} up to isomorphism. Define:

$$\phi^Z = \phi^{T^{*d}} \wedge (H(Hp \rightarrow p) \rightarrow (PHp \rightarrow Hp)) \wedge (G(Gp \rightarrow p) \rightarrow (FGp \rightarrow Gp))$$

We then have:

Theorem 3.1.1 $\mathbf{T} \models \phi^Z$ iff $\mathbf{T} \cong \mathbf{Z}$. □

This result is an immediate corollary of van Benthem’s THEOREM II.2.2.8 [5, page 63]. This theorem, a ‘best possible’ result concerning the definability of \mathbf{Z} in a Priorean

⁴Previously we only knew that $FFp \rightarrow Fp$ defined transitivity. In this case the uniform substitution of nominals for variables gives rise to a formula defining the same class, but this by no means always occurs as we know from the previous chapter. For example, p defines the empty class of frames, whereas i defines the class of frames of cardinality 1. Another example is provided by $p \rightarrow \neg Fp$. This defines the class of frames satisfying the condition $\forall xy(x \not\prec y)$, the totally disconnected frames. Replacing p uniformly with i gives a formula defining irreflexivity. Note, in keeping with the discussion of the Inheritance Lemma in the previous chapter, that the totally disconnected frames are a (proper) subclass of the irreflexive frames. In what follows we also refer to $FFi \rightarrow Fi$ as 4.

⁵As all discrete frames are irreflexive we don’t need to include I in the defining conjunction.

language, states that \mathbf{Z} is tense-logically definable on the class of *connected strict partial orders*. This means that if we restrict ourselves to considering the class of connected strict partial orders, then we can define \mathbf{Z} (up to isomorphism) in a Priorean language. van Benthem proves this by observing that if we conjoin the modified Löb axioms, the unboundedness axioms and:

$$Pp \rightarrow H(Fp \vee p \vee Pp), \text{ and } Fp \rightarrow G(Pp \vee p \vee Fp),$$

we narrow down the class of suitable frames to precisely \mathbf{Z} . Now the effect of the unboundedness axioms is clear, and the joint effect of the two modified Löb axioms on transitive frames is to demand that only a finite number of points can lie between any two points t and t' of the frame. What the last two axioms — known as the McTaggart axioms — secure is the condition ‘local linearity to the left’ (to the right):

$$\forall xyz(y < x \wedge z < x \rightarrow (z < y \vee z = y \vee y < z)).$$

That is, the McTaggart axioms do not define real trichotomy — nothing can in a Priorean language — just this local version. van Benthem’s result says that if we restrict our attention to the class of connected SPOs, there is only one frame on which these axioms can all be valid together, namely \mathbf{Z} . We need merely observe that using nominals we can restrict ourselves appropriately. The inclusion of \mathbf{I} and \mathbf{TRAN} among the conjuncts of $\phi^{\mathbf{Z}}$ restricts us to the class of SPOs; the inclusion of the \mathbf{TR} conjunct then ensures that we are restricted to a class of frames that is both connected and linear — namely the STOs — and the STOs are precisely the SPOs that are connected and validate the McTaggart axioms. Note, in passing, that as a result of this argument we can see that \mathbf{Disc} is not needed as a conjunct of $\phi^{\mathbf{Z}}$.

It is similarly straightforward to show that we can define \mathbf{N} up to isomorphism. Define $\phi^{\mathbf{N}}$ to be the conjunction of the following formulas: $Pi \vee i \vee Fi$, $F\top$, $H(Hp \rightarrow p) \rightarrow Hp$, and $G(Gp \rightarrow p) \rightarrow (FGp \rightarrow Gp)$. Then we have:

Theorem 3.1.2 $\mathbf{T} \models \phi^{\mathbf{N}}$ iff $\mathbf{T} \cong \mathbf{N}$ □

Once again this result is a simple corollary of a result of van Benthem’s [7, page 224] which states that the wff obtained from $\phi^{\mathbf{N}}$ by replacing \mathbf{Tr} in $\phi^{\mathbf{N}}$ by the conjunction

of the McTaggart axioms defines N on *connected frames*. Once again, in languages with nominals by using Tr instead of the McTaggart axioms we can force the desired trichotomy.

It is important to note that both ϕ^Z and ϕ^N are *mixed* sentences. By appealing to the later result that only L_0 expressible classes of frames are definable using purely nominal sentences, we know that no purely nominal sentence can uniquely define these structures; and further, van Benthem's results concerning the definability of these structures in Priorean languages are, in a precise sense that will become clear when we have discussed preservation results, 'best possible' results. The mixture of nominals and variables is thus necessary.

Having uniquely defined N , it is natural to ask whether all initial segments of this structure are similarly definable; once again, this can't be done in a purely Priorean language. They all are, and only nominals are needed. The proof of the following lemma is left to the reader.

Lemma 3.1.3 *Let $n \in N$ such that $n \geq 1$. Let $\phi^{T^n} = \phi^T \wedge G^n \perp \wedge (F^{n-1} \top \vee P F^{n-1} \top)$. Then $T \models \phi^{T^n}$ iff T is a STO of length exactly n . \square*

Although we have an absolute grip on the sizes of these frames, this is very much a grip on *ordinality*, not *cardinality*. (In fact, as we shall shortly see, the only class of frames of a given cardinality that can be defined is the class of frames of cardinality 1.) That we can count from 1 through to ω on the STOs depends crucially on the simple and regular nature of these frames' orderings; that is, the grip we have on these numbers is a structured grip.

In fact we can count other things using nominals, for example branches, that cannot be counted using only variables. That is, with the aid of nominals we can demand that any point in a frame has exactly n successors. We break the task of defining this condition into two subtasks: demanding that every point has at most n successors, which variables can do; and demanding that every point has at least n successors, for which nominals are required.

Firstly, in any language at all we can stipulate that any point has at most n successors, as the following encoding of the Pigeonhole Principle show:

Lemma 3.1.4 *Let $n \geq 1$. In any language of NTL the class of frames such that every point has at most n successors is defined by:*

$$\bigwedge_{1 \leq \alpha \leq n+1} Fa_\alpha \rightarrow \bigvee_{1 \leq \alpha \leq n; 2 \leq \beta \leq n+1; \alpha \leq \beta} F(a_\alpha \wedge a_\beta),$$

where the a_α are distinct atoms. □

However although the class of all frames with at least one successor — the right unbounded frames — can be defined in languages with nominals by either $F\top$ or $Gi \rightarrow Fi$, and in Priorean languages by either $F\top$ or $Gp \rightarrow Fp$, we cannot demand at least n branches for any n bigger than 1 in purely Priorean languages. With nominals we can. The condition that there be at least two successors is defined by:

$$F\top \wedge (Fi \rightarrow F\neg i);$$

the condition that there be at least three successors is defined by:

$$F\top \wedge (Fi \wedge Fj \rightarrow F(\neg i \wedge \neg j));$$

and in general we can define the condition that there be at least n successors by choosing $n - 1$ distinct nominals i_1, \dots, i_{n-1} and writing:

$$F\top \wedge ((Fi_1 \wedge \dots \wedge Fi_{n-1}) \rightarrow F(\neg i_1 \wedge \dots \wedge \neg i_{n-1})).$$

In effect we have chosen primitive names for $n - 1$ of the successors (the i_α are these names), and then manufactured a new quasi-name, $\neg i_1 \wedge \dots \wedge \neg i_{n-1}$. Due to the unique denoting property of nominals, if this new entity ‘names’ any points at all, it must name points distinct from the denotations of the i_α . The schema then works by insisting that if the other names denote, this strange new one does too. Note that if variables are uniformly substituted for nominals in this schema we don’t define ‘at least n successors’; variables don’t work like names. Indeed it is not just the purely Priorean formulas

obtainable from this schema that fail to define ‘at least n successors’; as we shall see when we discuss p-morphisms, we cannot write down any purely Priorean sentence defining this class of frames. We can demand at least one successor in Priorean languages, but that is the limit.

As a simple corollary of the discussion so far, in any language with nominals we can define, for any choice of $n \in \mathbb{N}$, the frames of cardinality n whose relation is universal — simply conjoin 4,T,B and the formula demanding exactly n successors. Again this special subclass is not Priorean definable.

This discussion of branching and counting could be generalised and extended in all sorts of directions; for example it should be clear that in languages with nominals we can control not only the number of successors of a given point, but the number of successors the successors of that point have, and so on. We won’t pursue this line however, but will content ourselves with observing that the following pretty classes of frames are definable: for any n we can define the class of oriented n -sided polygons.

Definition 3.1.2 *For all $n \geq 3$ define the oriented n -gon to be the frame whose underlying set is $\{0, \dots, n-1\}$, and whose relation is given by $m <_n m'$ iff $m+1 = m'$, where the addition is performed mod n . We say that a frame \mathbf{T} is an oriented n sided polygon iff $\mathbf{T} \cong n\text{-gon}$, for some $n \geq 3$. \square*

Now for any $n \geq 3$ we can define the n -sided polygons. Consider the case of squares. To define this we conjoin formulas defining irreflexivity, exactly one predecessor, exactly one successor, $i \vee Fi \vee FFi$ and $i \rightarrow FFFi$. The general case is left to the reader. The polygon example, the definability of finite universal relations, and the definability of the initial segments of \mathbb{N} all show that it is possible to get an absolute grip on some finite frames because of their regular path structure.

3.2 Preservation Results

Given that a formula ϕ is valid on some frame \mathbf{T} , are there general results guaranteeing the validity of ϕ on frames not isomorphic to \mathbf{T} ? There are four classic results of this kind for standard tense and modal logic: validity is preserved under the formation of generated subframes, disjoint unions, and p-morphic images; and anti-preserved under the formation of ultrafilter extensions. In this section we examine how the introduction of nominals affects these results.

Before examining these results in detail, let's consider the sort of information such preservation results give us. Firstly note that they can be construed as closure conditions on definable classes of frames: for example, the p-morphism result for Priorean languages expresses the fact that Priorean definable classes of frames are closed under the formation of p-morphic images. Secondly, and more importantly, such results are *negative* results concerning the expressive power of our languages. For example, the p-morphism result says that no formula ϕ can tell the difference between a frame \mathbf{T} and any p-morphic image of that frame, at least as far as validity is concerned: preservation results are the codification of methods that succeed in fooling formulas as to which frame they're being evaluated on. This shows itself in the typical applications of such theorems: they are commonly used to show that some condition is *not* definable.

Given that preservation results are negative results on expressive power, and given that mixed languages are more expressive than purely Priorean ones, we might expect that some of the earlier mentioned preservation theorems will be lost for languages with nominals. This does, in fact, occur. Only the generated subframe and ultrafilter extension results remain; the p-morphism and disjoint union results fail.

In this section we also gain two important model equivalence results; generated submodels: the latter result is rather interesting in view of the failure of the associated validity preservation result. Both equivalences are essential for later work.

3.2.1 Generated Subframes

By a (temporal) *generated subframe* $\mathbf{S} = \langle S, <_s \rangle$ of a frame $\mathbf{T} = \langle T, <_t \rangle$ is meant a subframe \mathbf{S} of \mathbf{T} such that:

$$\forall s \in S \forall t \in T ((s <_t t \vee t <_t s) \rightarrow t \in S)$$

That is, a generated subframe of \mathbf{T} is a subframe \mathbf{S} of \mathbf{T} whose underlying set S is closed under both the successor and predecessor relations of the original frame; that is, S is $<>$ -closed. If S is an arbitrary subset of T by the *subframe of \mathbf{T} generated by S* is meant the smallest generated subframe of \mathbf{T} to contain S . This generated subframe must always exist as \mathbf{T} will suffice as the smallest containing frame even if nothing else will. Note that if S is a singleton $\{t\}$, the subframe generated by S is connected — this is easily shown by induction and is crucial for the completeness results of the next two chapters.

For purely Priorean languages we have that if \mathbf{S} is a generated subframe of \mathbf{T} , then $\mathbf{T} \models \phi$ implies $\mathbf{S} \models \phi$ — no formula can tell the difference between a frame and any island of that frame. The result is usually proved by using the notion of a *generated submodel*:

Definition 3.2.1 $\langle \mathbf{S}, V^s \rangle$ is said to be a generated submodel of $\langle \mathbf{T}, V \rangle$ iff \mathbf{S} is a generated subframe of \mathbf{T} , and for all atoms a , $V^s(a) = V(a) \cap S$. \square

Now for Priorean languages it is obvious that given a model $\langle \mathbf{T}, V \rangle$ and a generated subframe \mathbf{S} , the function V^s defined by $V^s(a) = V(a) \cap S$ for all atoms a , is in $Val(\mathbf{S})$ and hence $\langle \mathbf{S}, V^s \rangle$ is a generated submodel of $\langle \mathbf{T}, V \rangle$ — we call it the generated submodel *induced by* $\langle \mathbf{T}, V \rangle$ and \mathbf{S} . A simple induction shows that for all $s \in S$:

$$\langle \mathbf{T}, V \rangle \models \phi[s] \text{ iff } \langle \mathbf{S}, V^s \rangle \models \phi[s].$$

This is the Generated Submodel Theorem for Priorean languages: the desired *frame preservation* result is an immediate corollary of this *model equivalence* result.

What about languages containing nominals? It should be intuitively clear that an analogous frame preservation result holds for them as well, and it seems sensible to derive it from a generated submodel equivalence result for languages with nominals. But note that for languages with nominals we cannot simply adopt the definition of generated submodels given above: given a model $\langle T, V \rangle$ and a generated subframe S of T , the function V^s defined by $V^s(a) = V(a) \cap S$ need *not* be a valuation on S : for if $V(i) \not\subseteq S$ then $V^s(i) = \emptyset$ and hence $V^s \notin \text{Val}(S)$. The moral is: $\langle S, V^s \rangle$ is not guaranteed to be a generated submodel of $\langle T, V \rangle$ — we must exercise more care in inducing submodels for languages with nominals. The following definitions pin down what is required.

Definition 3.2.2 Let $\langle T, < \rangle$ be a frame and $S \subseteq T$. A valuation $V \in \text{Val}(T)$ is said to be in S iff for all $i \in \text{NOM}$, $V(i) \subseteq S$. \square

Definition 3.2.3 Let S be a generated subframe of T . Let $\langle T, V \rangle$ be any model such that V is in S . Let $V \downarrow_S$ be the function:

$$V \downarrow_S : \text{ATOM} \longrightarrow \text{Pow}(S)$$

defined by $V \downarrow_S(a) = V(a) \cap S$. Then the pair $\langle S, V \downarrow_S \rangle$ is said to be the generated submodel in S induced by $\langle T, V \rangle$. As it should always be clear from context which subframe we are interested in, we normally suppress the subscript and just write $V \downarrow$. \square

The definition is satisfactory: the stipulation that V is in S means that all nominals are assigned singleton subsets of S , hence $V \downarrow$ is an NTL valuation in S , and thus the induced submodel is a model. Also note that this definition of induced generated submodel coincides with the usual one for purely Priorean languages.

Definition 3.2.4 Let T be a frame and S be a generated subframe of T . Define:

$$\text{In}_S : \text{Val}(T) \longrightarrow \text{Val}(S) \cup \{\dagger\}$$

by:

$$\text{In}_S(V) = \begin{cases} V \downarrow & \text{if } V \text{ is in } S \\ \dagger & \text{otherwise} \end{cases}$$

\square

Note that $\dagger \notin \text{Im}(In_S)$ iff $T = S$ iff In_S is injective. Also note that for all $V' \in \text{Val}(S)$, $In_S^{-1}[V'] \neq \emptyset$ — which simply means that any NTL valuation on S can be extended to an NTL valuation on T .

We can now state the Generated Submodel Theorem for an arbitrary language of NTL. (Note that the statement of this theorem includes the result for Priorean languages as a special case.) The proof is by induction on $\text{deg}(\phi)$.

Theorem 3.2.1 (Generated Submodel Theorem) *Let S be a generated subframe of T and $V' \in \text{Val}(S)$. Then for all wffs ϕ , all $s \in S$, and all $V \in In_S^{-1}[V']$, $V(\phi, s) = V'(\phi, s)$* □

Corollary 3.2.1 *If S is a generated subframe of T then $T \models \phi$ implies $S \models \phi$.*

Proof:

Suppose ϕ is falsifiable on S . That is, there is some $V' \in \text{Val}(S)$ and some $t \in S$ such that $V'(\phi, t) = -1$. But then by the previous theorem, for any $V \in In_S^{-1}(V')$, we have $V(\phi, t) = -1$. As $In_S^{-1}(V)$ is not empty we have $T \not\models \phi$. □

As a first application of this theorem we note that following simple condition is not definable: $\exists x(x < x)$. Suppose there was an NTL wff ϕ that defined this condition. Consider any frame that contains only two points t_1 and t_2 , such that $t_1 \not\prec t_2$, $t_2 \not\prec t_1$, $t_1 < t_1$, and $t_2 \not\prec t_2$. Then ϕ is valid on this frame because of t_1 . But by the previous theorem ϕ must remain valid on the generated subframe consisting of only the single irreflexive point t_2 — clearly impossible. Note also that the same frame and same argument also shows that the stronger condition:

$$\exists x((x < x) \wedge \forall y(y \neq x \rightarrow x \not\prec y \wedge y \not\prec x))$$

is not definable. That is, we cannot express the existence of an isolated reflexive point. This example will be useful to us when we discuss D logic.

As a second example we will prove the assertion made in the previous section that for any cardinal number $\kappa > 1$, the class of frames of cardinality κ is undefinable. This



is virtually immediate. For any $\kappa > 1$ consider any frame of cardinality κ whose relation is empty — that is, any totally disconnected frame of cardinality κ . Any totally disconnected frame of lower cardinality is (isomorphic to) a generated subframe of this frame, and thus shares the same NTL theory as the larger frame. Hence no wff ϕ distinguishing the larger cardinality can be found.

As a third application, let us consider connected frames and disconnected frames. The first result is easy: the class of disconnected frames is not NTL definable. For suppose ϕ is a wff that defines this condition, and let \mathbf{T} be a disconnected frame. Let t be any member of T , and let S be the smallest generated subframe of \mathbf{T} containing t . By the generated subframe preservation result, $S \models \phi$. As S is connected there can be no such ϕ .

The next result is more interesting: the class of connected frames is not definable either. As we shall see in the next section, this result is a virtual triviality for purely Priorean languages: but for languages with nominals things are slightly more tricky. Actually, by appealing to the result that the only conditions a purely nominal language can define are L_0 expressible, we have that no pure language can define the class. But couldn't some suitable mixed sentence accomplish the task? The answer is no: and we deduce this result as a corollary of the following frame comparison ⁶ between $\mathbf{N}^S = \langle N, S \rangle$ and $\mathbf{N}^S \uplus \mathbf{N}^S$. The former frame is the natural numbers under the successor relation — that is, $\langle n, m \rangle \in S$ iff $m = n + 1$. The latter frame is the disjoint union of the first frame with itself. (A general definition of the disjoint union of frames is given in the next section. For present purposes it suffices that the latter frame contains precisely two generated subframes, each of which is isomorphic to $\langle N, S \rangle$, and that this frame is disconnected. In particular, no point in one generated subframe is related to any point in the other.)

Theorem 3.2.2 $Th(\mathbf{N}^S) = Th(\mathbf{N}^S \uplus \mathbf{N}^S)$

⁶Johan van Benthem suggested this frame comparison.

Proof:

One inclusion is immediate: suppose $\phi \in Th(N^S \uplus N^S)$. As N^S is (isomorphic to) a generated subframe of $N^S \uplus N^S$, by the generated subframe preservation result, $\phi \in Th(N^S)$.

The other direction is pretty. Suppose $\phi \notin Th(N^S \uplus N^S)$. Let V be any valuation on this frame that falsifies ϕ at t . If we knew V was *in* the generated subframe of $N^S \uplus N^S$ that t is in, we could use the generated submodel result to transfer this falsifying valuation to the smaller frame, N^S . Now we have no guarantee that V is in this generated subframe — but using the Horizon Lemma we can construct a V' falsifying ϕ that *is* in the generated subframe containing t . Proceed as follows. Suppose $td(\phi) = n$. Then as the relation on $N^S \uplus N^S$ is intransitive, $t \not\prec t + n + 1$. So simply let V' be the valuation obtained from V by letting all the nominals that V sends to other generated subframe denote $t + n + 1$; other than this, $V' = V$. By design, this new valuation is in the subframe of interest. And by design V' and V satisfy the requirements of the Horizon Lemma, and hence V' falsifies ϕ at t . Transfer V' to the smaller frame and we are through. \square

Thus there can be no ϕ valid on precisely the connected frames: N^S is connected and $N^S \uplus N^S$ is disconnected and their NTL theories are identical. This example also shows that $Th(\mathbf{T}) = Th(\mathbf{T}')$ does not imply $\mathbf{T} \cong \mathbf{T}'$.

Analogous results hold for languages of NML. In fact the only change that needs to be made is in the definition of generated subframe: for modal languages we do not require that the set underlying a generated subframe be closed under both successor and predecessor, only that it be closed under successor. This reflects the fact that in tense logics we can look both forwards along $<$ (using F), and back (using P); whereas in modal logics we can only look forwards (using \Diamond). With this single change made, all the above definitions are suitable, and all the results hold, for languages of NML.

We can use the fact that the validity of modal formulas is preserved under the formation of modal generated subframes to show that any language of NTL can define a condition that no language of NML can. (To put the matter slightly inaccurately: lan-

languages of tense logic are more expressive than those of modal logic.⁷) Consider the condition: $\forall x \exists y (y < x)$. This can be defined by either $p \rightarrow PFp$ or $i \rightarrow PFi$. But no formula ϕ of a modal language can define this condition. Suppose there was such a ϕ . Then $Z \models \phi$. But N is a modal generated subframe of Z , and ϕ must be false on this frame at 0, contradicting the modal generated subframe preservation result. The general question of which conditions are definable in NML and which require NTL — especially with regard to purely nominal formulas — is an interesting one, and will arise naturally when we consider interval based languages in the final chapter. I briefly mention some matters that bear on this issue at the end of this chapter.

3.2.2 Disjoint Unions

Let $\{T_m : m \in M\}$ be a non-empty family of frames. For all $m \in M$ we define T'_m to be $\langle T'_m, <'_m \rangle$ where:

$$T'_m = \{\langle m, t \rangle : t \in T_m\}$$

and

$$<'_m = \{\langle \langle m, t_1 \rangle, \langle m, t_2 \rangle \rangle : \langle t_1, t_2 \rangle \in <_m\}.$$

By the disjoint union $\cup\{T_m : m \in M\}$ is meant the frame

$$\langle \bigcup\{T'_m : m \in M\}, \bigcup\{<'_m : m \in M\} \rangle.$$

As for all $m \in M$ $T'_m \cong T_m$, in what follows we usually ignore the distinction and just talk of T_m .

For Priorean languages we have the following preservation result:

⁷The inaccuracy is that it hasn't been stated what sort of atoms these languages contain: a modal language containing nominals can express conditions that a purely Priorean language cannot and vice-versa. What we can say is that given a language of NML and a language of NTL each of which has the same number of each sort of atom, then the modal language is less expressive than the tensed language.

Theorem 3.2.3 *If for all $m \in M$ $\mathbf{T}_m \models \phi$, then $\bigcup \mathbf{T}_m \models \phi$.* \square

That is, if a formula is valid on each frame in some collection of frames, it remains valid on the frame created by considering that collection as a single entity. This follows as a result of the generated submodel result for Priorean languages; see [5, page 147] for details.

An immediate consequence of this result is that Priorean languages cannot define the universal relation $\forall xy(x < y)$. Suppose some wff ϕ of a Priorean language does define this condition. Take any two frames \mathbf{T}_1 and \mathbf{T}_2 whose relation is universal; ϕ is valid on each. By the above result, ϕ must remain valid on $\mathbf{T}_1 \cup \mathbf{T}_2$ — but this frame's relation is not universal.

Note that it also yields a proof of the non-definability of connectedness by Priorean languages that is virtually immediate. The disjoint union preservation result can be paraphrased as: any Priorean definable class of frames is closed under the formation of disjoint unions. But the class of connected frames is clearly not so closed: a disjoint union of frames produces a frame consisting of separated islands — precisely what it's designed to do! Note that this paraphrase of the preservation result actually gives us something stronger: no non-empty Priorean definable class of frames contains only connected frames.

The situation changes for languages containing nominals; the preservation theorem no longer holds. An immediate counterexample is given by the class of trichotomous frames, defined by $Pi \vee i \vee Fi$. Take any two members of this class. The defining formula is valid on each. But the defining formula fails on the disjoint union of the two members, as the disjoint union consists of two separated 'parallel lines', and hence is not trichotomous.⁸

⁸It should now be clear why van Benthem's result on the definability of \mathbf{N} in Priorean languages is 'best possible'. His result shows that \mathbf{N} can be defined up to isomorphism to 'within one preservation result'. That is, only the fact that the validity of Priorean sentences is preserved under the formation of disjoint unions prevents its exact definition. As will shortly become clear,

Another counterexample is provided by the class of directed frames. Consider the two islands in the disjoint union of two directed frames. No two points from distinct islands are directed, hence the disjoint union is not directed. Yet another counterexample is provided by the universal relation; this condition is definable using nominals, by $\forall i$. Note something common to these counterexamples: each member of these definable classes is a connected frame. This is important: in languages with nominals, some non-empty classes of frames consisting solely of connected frames, are definable.

We now turn to a rather pretty result. Reflection shows that disjoint union preservation only just fails for languages containing nominals. Suppose we have two frames \mathbf{T}_1 and \mathbf{T}_2 on each of which ϕ is valid. To keep things simple suppose ϕ contains occurrences of only one nominal, say i . We know that we cannot conclude that $\mathbf{T}_1 \uplus \mathbf{T}_2 \models \phi$, but why not? The reason is that in any valuation on $\mathbf{T}_1 \uplus \mathbf{T}_2$, on one of the components, say \mathbf{T}_1 , i will be false everywhere. This is a situation that the validity of ϕ on the component frames simply gives us no information about: in any valuation on either frame i is true *somewhere*.

But suppose we knew something more: namely that not only was ϕ valid on each frame, but $\phi[\perp/i]$ was also. Then, intuitively, we *would* have the information needed to guarantee validity on the disjoint union: the validity of the new formula blocks the possibility that i being false everywhere in a component will cause trouble. We will shortly demonstrate that this is the case. Indeed we show something stronger: not only is the condition *sufficient*, it is also *necessary* as long as the disjoint union is not trivial — that is, as long as at least two frames are stuck together.

To state and prove the theorem in full generality we need to extend the above intuitions to the case where ϕ contains many different nominals. Essentially all we need to do is account for all the different ways the nominals can be ‘dealt out’ — like cards

this result concerning the definability of \mathbf{Z} shows that this structure can be uniquely defined to ‘within *two* preservation results.’ Here the formation of both disjoint unions and p-morphic images prevents an exact definition.

from a pack — to the ‘players’ — the components of the disjoint union.⁹ That is, we must take into account all possible uniform substitutions of \perp for nominals in ϕ . This is simple, and motivates the following definitions:

Definition 3.2.5 *Let a be an arbitrary atom. We define a function:*

$$Sub^\perp : Pow(ATOM) \times WFF \longrightarrow WFF$$

by:

$$\begin{aligned} Sub^\perp(X, a) &= \begin{cases} \perp & \text{if } a \in X \\ a & \text{if } a \notin X \end{cases} \\ Sub^\perp(X, \neg\phi) &= \neg(Sub^\perp(X, \phi)) \\ Sub^\perp(X, \phi \wedge \psi) &= Sub^\perp(X, \phi) \wedge Sub^\perp(X, \psi) \\ Sub^\perp(X, F\phi) &= F(Sub^\perp(X, \phi)) \\ Sub^\perp(X, P\phi) &= P(Sub^\perp(X, \phi)) \end{aligned}$$

□

Definition 3.2.6 *For any non-empty family of frames $\{T_m : m \in M\}$ we define:*

$$Out : \{T_m : m \in M\} \times Val(\biguplus T_m) \longrightarrow Pow(NOM)$$

by:

$$Out(T_n, V) = \{i \in NOM : V(i) \not\subseteq T_n\}$$

□

Although this last definition looks rather cumbersome, *Out* actually does something rather simple: it takes a component of a disjoint union and a valuation and returns all the nominals which denote a point outside that component. The next lemma gives the reason for introducing Sub^\perp and *Out*. In essence it says that nominals whose denotation lies outside the component containing the point of evaluation, behave like \perp .

⁹A particular deal, of course, is just a valuation.

Lemma 3.2.1 *Let some non-empty family of frames $\{\mathbf{T}_m : m \in M\}$ be given, and let \mathbf{T}_n be an arbitrary member of this family. Then for all $\phi \in WFF$, for all $V \in Val(\biguplus \mathbf{T}_m)$ and for all $t \in \mathbf{T}_n$ we have:*

$$\langle \biguplus \mathbf{T}_m, V \rangle \models \phi[t] \text{ iff } \langle \biguplus \mathbf{T}_m, V \rangle \models Sub^\perp(Out(\mathbf{T}_n, V), \phi)[t]$$

Proof:

By induction on $deg(\phi)$. Suppose ϕ is a nominal, say i . Then if $i \notin Out(\mathbf{T}_n, V)$, $Sub^\perp(Out(\mathbf{T}_n, V), i) = i$ and the above equivalence holds trivially. (Note that this argument immediately gives us the base case involving variables as well: no variable is ever in any set in the image of Out .) So suppose $i \in Out(\mathbf{T}_n, V)$. Then $Sub^\perp(Out(\mathbf{T}_n, V), i) = \perp$. Obviously for all $t \in \mathbf{T}_n$, $\langle \biguplus \mathbf{T}_m, V \rangle \not\models \perp[t]$. But $i \in Out(\mathbf{T}_n, V)$ means $V(i) \not\subseteq \mathbf{T}_n$, hence for any $t \in \mathbf{T}_n$ we also have $\langle \biguplus \mathbf{T}_m, V \rangle \not\models i[t]$.

The inductive cases are straightforward, following immediately from the definition of Sub^\perp . \square

Definition 3.2.7 *Let $Nom(\phi) = \{i \in NOM : i \text{ occurs in } \phi\}$. Then for any wff ϕ , we define $S^\perp(\phi)$ to be: $\{Sub^\perp(X, \phi) : X \subseteq Nom(\phi)\}$.* \square

Note that $S^\perp(\phi)$ is never empty, as for all wffs ϕ , $\emptyset \subseteq Nom(\phi)$, hence we have $Sub^\perp(\emptyset, \phi) = \phi$, and thus ϕ is always in $S^\perp(\phi)$. Note also that $S^\perp(\phi)$ is always a finite set of wffs. For any wff ϕ we define ϕ^\perp to be the conjunction of all wffs in $S^\perp(\phi)$ if this set is not a singleton; and ϕ otherwise. For example, let ϕ be the formula $Fi \rightarrow Pj$. Then $S^\perp(\phi)$ is the set:

$$\{Fi \rightarrow Pj, F\perp \rightarrow Pj, Fi \rightarrow P\perp, F\perp \rightarrow P\perp\}$$

and ϕ^\perp is the formula:

$$(Fi \rightarrow Pj) \wedge (F\perp \rightarrow Pj) \wedge (Fi \rightarrow P\perp) \wedge (F\perp \rightarrow P\perp)$$

Clearly $S^\perp(\phi)$ and ϕ^\perp encode ‘deals’ in the required fashion. We can now state and prove the theorem.

Theorem 3.2.4 *Let $\{\mathbf{T}_m : m \in M\}$ be a family of frames such that $\text{card}(M) \geq 2$. Then for all wffs ϕ we have:*

$$\biguplus \mathbf{T}_m \models \phi \text{ iff } \forall m \in M \ \mathbf{T}_m \models S^\perp(\phi)$$

Proof:

(\Rightarrow). We show the contrapositive. Suppose for some $n \in M$, \mathbf{T}_n falsifies $S^\perp(\phi)$. That is, for some $t \in T_n$, $V \in \text{Val}(\mathbf{T}_n)$, and $\sigma \in S^\perp(\phi)$ we have:

$$\langle \mathbf{T}_n, V \rangle \not\models \sigma[t]$$

By the Generated Submodel Theorem we can extend V to a valuation $V' \in \text{Val}(\biguplus \mathbf{T}_m)$ such that

$$\langle \biguplus \mathbf{T}_m, V' \rangle \not\models \sigma[t].$$

(To use the terminology of the proof of the Generated Submodel Theorem, simply let V' be any member of $\text{In}_{\mathbf{T}_n}^{-1}[V]$. Note, in passing, that $\text{Out}(\mathbf{T}_n, V') = \emptyset$.)

Obviously $\text{Nom}(\sigma) \subseteq \text{Nom}(\phi)$. Now if $\text{Nom}(\phi) = \text{Nom}(\sigma)$ then $\phi = \sigma$ and we are through. So suppose $\text{Nom}(\phi) \setminus \text{Nom}(\sigma) \neq \emptyset$. Choose an arbitrary $t' \notin T_n$: that such a t' exists is guaranteed by our assumption that $\text{card}(M) \geq 2$, and obviously $t \neq t'$. Define V^n by:

$$\begin{aligned} V^n(i) &= \begin{cases} \{t'\} & \text{if } i \in \text{Nom}(\phi) \setminus \text{Nom}(\sigma) \\ V'(i) & \text{otherwise} \end{cases} \\ V^n(p) &= V(p) \text{ for all } p \in \text{VAR} \end{aligned}$$

As V^n and V' agree on all atoms in σ , by the Agreement Lemma we have:

$$\langle \biguplus \mathbf{T}_m, V^n \rangle \not\models \sigma[t]$$

But as:

$$\text{Out}(\mathbf{T}_n, V^n) = \{i : i \in \text{Nom}(\phi) \setminus \text{Nom}(\sigma)\}$$

we have:

$$\text{Sub}^\perp(\text{Out}(\mathbf{T}_n, V^n), \phi) = \sigma$$

and thus by the previous lemma:

$$\langle \biguplus \mathbf{T}_m, V^n \rangle \not\models \phi[t]$$

and we have established the contrapositive of the desired result.

(\Leftarrow). Again we argue by contrapositive. Suppose there is a point $t \in \biguplus \mathbf{T}_m$ and a valuation V on that frame such that:

$$\langle \biguplus \mathbf{T}_m, V \rangle \not\models \phi[t]$$

for some $n \in M$ and $t \in \mathbf{T}_n$. Let $\psi = Sub^\perp(Out(\mathbf{T}_n, V), \phi)$. Then by the previous lemma we have:

$$\langle \biguplus \mathbf{T}_m, V \rangle \not\models \psi[t]$$

We now construct a new valuation V^n . The essential point to note about this function is that it is in \mathbf{T}_n .

$$\begin{aligned} V^n(i) &= \begin{cases} V(i) & \text{if } i \in Nom(\psi) \\ \{t\} & \text{otherwise} \end{cases} \\ V^n(p) &= V(p) \text{ for all } p \in VAR \end{aligned}$$

As V^n and V agree on all atoms in ψ we have that:

$$\langle \biguplus \mathbf{T}_m, V^n \rangle \not\models \psi[t]$$

But as V^n is in \mathbf{T}_n by our generated submodel result we have:

$$\langle \mathbf{T}_n, V^n \rangle \models \psi[t]$$

It only remains to verify that ψ is a wff in $S^\perp(\phi)$. But this is indeed the case for ψ by definition is $Sub^\perp(Out(\mathbf{T}_n, V), \phi)$, and inspection of the inductive definition of Sub^\perp reveals that:

$$Sub^\perp(Out(\mathbf{T}_n, V), \phi) = Sub^\perp(Out(\mathbf{T}_n, V) \cap Nom(\phi), \phi)$$

and this last sentence must be in $S^\perp(\phi)$ as $Out(\mathbf{T}_n, V) \cap Nom(\phi) \subseteq Nom(\phi)$. Hence we have shown the contrapositive of the required result. \square

It is immediate from the definition of ϕ^\perp and the previous theorem that if $\{\mathbf{T}_m : m \in M\}$ is a family of frames such that $\text{card}(M) \geq 2$, then for all wffs ϕ we have that $\bigcup \mathbf{T}_m \models \phi$ iff $\forall m \in M \mathbf{T}_m \models \phi^\perp$.

Note that weak nominal validity is preserved under the formation of disjoint unions.

3.2.3 P-morphisms

In this section we examine truth and validity preserving maps between structures. The fundamental concept needed is that of a (temporal) *p-morphism* between frames:

Definition 3.2.8 (Frame p-morphisms) Let $\mathbf{S} = \langle S, <_s \rangle$ and $\mathbf{T} = \langle T, <_t \rangle$ be frames and $f : S \rightarrow T$ be such that:

1. $s <_s s'$ implies $f(s) <_t f(s')$
2. $f(s) <_t t$ implies there is an $s' \in S$ such that $s <_s s'$ and $f(s') = t$
3. $t <_t f(s)$ implies there is an $s' \in S$ such that $s' <_s s$ and $f(s') = t$.

Then f is a p-morphism from \mathbf{S} to \mathbf{T} . Given two frames \mathbf{S} and \mathbf{T} , if there exists a p-morphism from \mathbf{S} onto \mathbf{T} we say that the frame \mathbf{T} is a p-morphic image of the frame \mathbf{S} . □

Note that p-morphisms preserve more structure than ordinary order homomorphisms (these demand only clause 1 and thus don't back preserve $<_t$), and less than strong homomorphisms (which demand that clause 1 be a biconditional and thus totally back preserve $<_t$). We sometimes say that p-morphisms partially back preserve $<_t$. Modal p-morphisms are defined by dropping clause 3.

Surjective p-morphisms preserve validity for Priorean languages; they do not do so for languages with nominals, and counterexamples abound. The unique function from \mathbf{Z} to the singleton reflexive frame $\langle 0, \{\langle 0, 0 \rangle\} \rangle$ is a surjective p-morphism. For Priorean languages this yields an immediate proof that neither irreflexivity nor asymmetry is definable: the source frame \mathbf{Z} has both characteristics, the target frame has neither,

and Priorean validity is transmitted via p -morphisms. For languages with nominals on the other hand, this example shows that p -morphisms don't preserve validity: both $i \rightarrow \neg Fi$ and $i \rightarrow \neg FFi$ are valid on \mathbb{Z} and invalid on the singleton reflexive loop. Another counterexample follows from the fact that with nominals we can define the class of universal relations of any fixed finite cardinality n . The unique contraction down to the singleton reflexive loop is a surjective p -morphism — indeed, a surjective strong homomorphism — showing that even surjective strong homomorphisms don't preserve nominal validity. A pretty p -morphism is constructed in [5, pages 160-161]. The source frame is discrete, the target frame indiscrete, thus demonstrating that discreteness is not Priorean definable. As discreteness is definable with nominals, van Benthem's construction is yet another counterexample. Finally, consider n branching trees of depth ω . All points in such trees have precisely n -successors, and we know that for all $n \in \mathbb{N}$ we can write an expression in nominals valid on all frames with branch factor n . But the mapping from n branching trees of depth ω to the natural numbers under the successor relation, \mathbb{N}^S , which associates with each node its depth is a surjective p -morphism, and thus for all $n \geq 2$ we have an example of the non-transmission of nominal validity. The loss of this preservation result in such thorough going fashion is the most dramatic evidence we have of how differently languages with nominals and purely Priorean languages encode relational conditions.

We now examine p -morphisms between our other type of semantic structure: models.

Definition 3.2.9 (Model p -morphisms) *Let S and T be frames, f be a p -morphism from S to T , and V_s and V_t be valuations on S and T respectively such that for all atoms a :*

$$s \in V_s(a) \text{ iff } f(s) \in V_t(a).$$

Then f is a p -morphism from the model $\langle S, V_s \rangle$ to the model $\langle T, V_t \rangle$. Given two models M_s and M_t , if there exists a p -morphism from M_s to M_t then we say that M_t is a p -morphic image of M_s ; or simply that M_s and M_t are a p -morphic pair of models. \square

Even for languages with nominals, p -morphic pairs of models are equivalent. More precisely, just as for Priorean languages we can prove the following lemma by induction.

As usual, only the base case involving nominals is new and this follows immediately from the definition of model p-morphisms.¹⁰

Lemma 3.2.2 *Let f be a p-morphism from $M_s = \langle S, V_s \rangle$ to $M_t = \langle T, V_t \rangle$. Then for all $s \in S$ and all wffs ϕ , $M_s \models \phi[s]$ iff $M_t \models \phi[f(s)]$. \square*

While this result is extremely useful (the completeness results in Chapter 5 hinge on it), upon reflection it is not particularly surprising. Establishing a p-morphic link between two *models* involves establishing two quite distinct things. Firstly, it must be shown that the two underlying frames are in the correct structural relation: one must be a p-morphic image of the other. Secondly, it must be shown that the valuations on the source and target frames ‘mesh’ correctly with the p-morphism. Now when working with Priorean languages, given two p-morphic frames it is very easy when presented with a model based on the target frame to construct a p-morphic model on the source: we just suck the target frame valuation back through the p-morphism to create a correctly meshing valuation on the source frame. (This is why surjective p-morphisms preserve Priorean validity: given a falsifying valuation on the target frame we can always form a falsifying valuation on the source frame by this method.) Ultimately our ability to do this stems from the fact that there are no constraints on the denotations of variables; we can assign any subsets we please and still have a valuation. Matters are not so straightforward with nominals: the existence of a frame p-morphism does not guarantee the existence of any p-morphic models at all. Two frames may be correctly structurally linked without it being possible to correctly mesh any pair of source and target frame valuations: the constraint that nominals be assigned singletons may prevent this. Thus a real difficulty is involved; what the above lemma tells us is that *once this difficulty has*

¹⁰The following lemma can be strengthened in a standard way. Following van Benthem [9, page 12] we can define the relational link between models known as a zigzag connection, and it is straightforward to show by induction that if two models M_1 and M_2 are zigzag related by R then $M_1 \models \phi[t_1]$ iff $M_2 \models \phi[t_2]$ for all wffs ϕ and all points t_1, t_2 such that $t_1 R t_2$.

been resolved and two *p*-morphic models have been found, things work as with Priorean languages. Two put the matter another way; both nominals and variables see *p*-morphic pairs of *models* the same way; it's just that there are fewer *p*-morphic pairs of models for languages with nominals.

This *p*-morphic model result is totally general: *p*-morphic models must be equivalent no matter what sorts we use. For example, in Chapter 6 interval nominals are introduced. It doesn't matter for present purposes what constraints this sort obeys; but given two *p*-morphic ^{models} which obey these constraints (whatever these may be), the models are equivalent. The hard part is making *p*-morphic models in the first place.

Reverting to validity, the notion of a validity preserving function between frames is an important one, but neither *p*-morphisms nor strong homomorphisms suffice. Is there anything we can say about the situation? It is at least possible to state a sufficient condition on *p*-morphisms which guarantees they preserve the validity of certain wffs ϕ . First we need the following definition:

Definition 3.2.10 *Let f be a *p*-morphism from S to T , and $n \geq 0 \in N$. We say that f is n -separating iff for all $s, s' \in S$, if $s \neq s'$ and $f(s) = f(s')$ then there is no path between s and s' of length less than or equal to n .* \square

Lemma 3.2.3 *Let ϕ be a wff such that $td(\phi) = n \geq 1$, and f a $2n + 1$ -separating surjective *p*-morphism from S to T . Then $S \models \phi$ implies $T \models \phi$.*

Proof:

Suppose $T \not\models \phi$. Then for some valuation $V_t \in Val(T)$ and $t \in T$, $V_t(\phi, t) = -1$. Let V_s be the function with domain $ATOM_{\mathcal{L}}$ and range $Pow(s)$ defined by $V_s(a) = [f^{-1}(V_t(a))]$, for all atoms a . If f is an isomorphism we are through; so suppose f is not injective. Then V_s need not be a \mathcal{L} valuation; nonetheless it is a \mathcal{L}' valuation where \mathcal{L}' is the purely Priorean language whose variables are $VAR_{\mathcal{L}} \cup NOM_{\mathcal{L}}$. (In short, we're temporarily pretending that nominals are variables.) Let s be some fixed member of $[f^{-1}(t)]$. Then using the the proof of the *p*-morphism Lemma we see that $V_s(\phi, s) = -1$,

where ϕ is now regarded as a \mathcal{L}' wff. We now use the fact that f is $2n + 1$ -separating and the Horizon Lemma to construct a \mathcal{L} valuation that falsifies ϕ on S at s .

As f is not injective there exist distinct points $s_1, s_2 \in S$ such that $f(s_1) = f(s_2)$. However there cannot be two such points in $Ker_n(s)$, for then there is a path of length $2n$ between s and s' contradicting the assumption that f is $2n + 1$ separating. So there is at least one point of S that lies outside $Ker_n(s)$; call this point s' . Define a new function V'_s from $ATOM_{\mathcal{L}}$ to $Pow(S)$ by:

$$\begin{aligned} V'_s(p) &= V_s(p) \text{ for all variables } p \\ V'_s(i) &= \begin{cases} \{s'\} & \text{if } V_s(i) \cap Ker_n(s) = \emptyset \\ V_s(i) \cap Ker_n(s) & \text{otherwise} \end{cases} \end{aligned}$$

Clearly V'_s is a \mathcal{L}' valuation, and by the Horizon Lemma: $V'_s(\phi, s) = -1 = V_s(\phi, s)$. But it is easy to show that if $V(s) \cap Ker_n(S)$ is not empty, it is a singleton. For suppose it contains distinct points s_1 and s_2 . Then by definition of V_s , f maps both points to the singleton $V_t(i)$; but as both s_1 and s_2 are in $Ker_n(s)$ there is a path between them of length at most $2n$, contradicting the fact that f is $2n + 1$ separating. Thus V'_s is an NTL valuation for \mathcal{L} and we are through. \square

Apart from this observation however, matters are unclear to me. Nonetheless, because the notion of maps between frames that preserve some aspects of structure and transmit validity — or even just the validity of certain special subclasses of wffs — is such a fundamental one, I am investigating a number of ideas that may give further insight. Some of these are noted later.

3.2.4 Ultrafilter extensions

The last of the four classic preservation results is the antipreservation of validity under the formation of *ultrafilter extensions*, an algebraically motivated construction. The addition of nominals does not affect this result. Because the part of the proof specific to nominals is virtually immediate — essentially it boils down to the fact that singletons of frames give rise to principal ultrafilters — and as the details of the non-nominal parts

are available elsewhere, (for example, [6, pages 32–34] and [38, pages 57–61]), only proof sketches are given. It is assumed that the reader is familiar with filters and ultrafilters, and their basic properties; Chang and Keisler contains everything required [18, pages 164–167]. The definition given in Chapter 2 of ‘algebraic valuation’ is used here.

Definition 3.2.11 (Ultrafilter extensions of frames) *Let $\mathbf{T} = \langle T, < \rangle$ be a frame. By $ue(\mathbf{T})$, the ultrafilter extension of \mathbf{T} , is meant the frame $\langle T_u, <_u \rangle$, where T_u is the set of all ultrafilters on T , and $<_u$ is defined by:*

$$U <_u U' \text{ iff } \forall S \subseteq T (S \in U' \Rightarrow \pi_f(S) \in U),$$

for all $U, U' \in T_u$. □

Note that $<_u$ is defined purely in terms of future projections; however this ordering gives us everything we desire, as the following equivalences testify:

Lemma 3.2.4 (Algebraic Order Equivalence Lemma) *For all $U, U' \in T_u$, $U <_u U'$ is equivalent to any of the following conditions:*

1. $\forall S \subseteq T (S \in U \Rightarrow \pi_p(S) \in U')$
2. $\forall S \subseteq T (\pi_g(S) \in U \Rightarrow S \in U')$
3. $\forall S \subseteq T (\pi_h(S) \in U' \Rightarrow S \in U).$

Proof:

Mimic the proof of the Order Equivalence Lemma given in the next chapter, which is essentially the transcription of the above result into tense logical syntax. At certain points in that proof two theorems of minimal tense logic are appealed to, namely $FH\phi \rightarrow \phi$ and $PG\phi \rightarrow \phi$. Their algebraic counterparts are $\pi_f(\pi_h(S)) \subseteq S$ and $\pi_p(\pi_g(S)) \subseteq S$. Establishing that these containments hold merely involves checking the definitions. With this noted, everything proceeds mechanically. □

Definition 3.2.12 (Ultrafilter extensions of models) *Let $\mathbf{M} = \langle \mathbf{T}, V \rangle$ be a model. By $ue(\mathbf{M})$, the ultrafilter extension of \mathbf{M} , is meant the model $\langle ue(\mathbf{T}), ue(V) \rangle$ where $ue(V)$*

is the valuation defined by:

$$ue(V)(a) = \{U \in ue(\mathbf{T}) : V(a) \in U\}$$

for all atoms a . □

Of course it needs to be checked that this definition makes sense for nominals — but this is immediate. Any singleton subset $\{t\}$ of T is contained in precisely one ultrafilter on T , namely the principal ultrafilter generated by $\{t\}$. (There clearly is at least one ultrafilter containing $\{t\}$, as trivially $\{t\}$ has the finite intersection property. The assumption that there are two distinct ultrafilters containing $\{t\}$ leads to immediate contradiction.) Thus $ue(V)$ really is a valuation on $ue(\mathbf{T})$ and thus $ue(\mathbf{M})$ is a model. We now have:

Lemma 3.2.5 *Let $\mathbf{M} = \langle \mathbf{T}, V \rangle$ be any model. Then for all wffs ϕ :*

$$ue(\mathbf{M}) \models \phi[U] \text{ iff } V(\phi) \in U.$$

Proof:

van Benthem [6, page 33] gives a proof by induction on $deg(\phi)$ for languages of modal logic with \Box taken as primitive. By appealing to the Algebraic Order Equivalence Lemma, both to move from his universal \Box primitive to our existential F and P primitives, and to add the backward looking clause for P to the induction, we see that his proof works for tensed languages as well. □

Corollary 3.2.2 *For any frame \mathbf{T} and any wff ϕ , $ue(\mathbf{T}) \models \phi$ implies $\mathbf{T} \models \phi$.*

Proof:

Suppose that for some valuation V on \mathbf{T} and some $t \in T$, that $\langle \mathbf{T}, V \rangle \not\models \phi[t]$. Then, using the algebraic notion of valuation, $t \notin V(\phi)$. Let U be the principal ultrafilter generated by $\{t\}$. Clearly $V(\phi) \notin U$, and so by the previous lemma, $\langle ue(\mathbf{T}), ue(V) \rangle \not\models \phi[U]$. □

For further discussion of ultrafilter extensions we refer the reader to [10, pages 20–26]. Here we will be content to give an example of something not definable in any language of NTL because this preservation result holds. The example, from [6, page 33], is $\forall t \exists t' (t < t' \wedge t' < t')$. This is not definable in any language of NTL because it is valid on $ue(\mathbb{N})$, but invalid on \mathbb{N} . With the remark that all the above definitions and results go through essentially unchanged for languages of NML, we close our discussion of ultrafilter extensions.

3.3 Some correspondences

In this section we compare languages of NTL with other languages for talking about frames. Firstly we compare them with three classical languages, L_0 , L_1 and L_2 ; then we compare them with standard languages of tense logic augmented by the D operator.

3.3.1 L_0 , L_1 and L_2

We have already met L_0 . L_1 is ^afirst order extension of L_0 , and L_2 a second order extension of L_0 — in fact it is essentially L_1 viewed as a second order language. Their relevance to standard tensed languages is as follows: as far as truth in a model is concerned, standard tense logical wffs correspond to certain L_1 wffs; whereas as far as validity on a frame is concerned, standard tense logical wffs correspond to L_2 wffs — though in certain interesting cases, as we already know, standard tense logical validity may correspond to the validity of L_0 wffs.

Let's be more specific. Suppose \mathcal{L} is a purely Priorean language. Let L_1 be L_0 augmented by a distinct new one place predicate P for every $p \in VAR_{\mathcal{L}}$. (Or, regard each $p \in VAR_{\mathcal{L}}$ as a one place predicate symbol of L_1 , and when it is being so regarded write p as P .) L_1 has a countably infinite supply of individual variables. Pick one of these and call it t_0 ; this reserved variable will play the role of the point of evaluation. Following [5, page 151] we now define the *standard translation* of \mathcal{L} wffs into

L_1 wffs:

$$\begin{aligned}
 ST(p) &= Pt_0 \\
 ST(\neg\phi) &= \neg ST(\phi) \\
 ST(\phi \wedge \psi) &= ST(\phi) \wedge ST(\psi) \\
 ST(F\phi) &= \exists x(t_0 < x \wedge [x/t_0]ST(\phi)) \\
 ST(P\phi) &= \exists x(x < t_0 \wedge [x/t_0]ST(\phi))
 \end{aligned}$$

In the last two clauses $[x/t_0]ST(\phi)$ is the result of substituting some individual variable x that hasn't yet occurred into the standard translation of ϕ .

The point of the translation is this: a model $\langle T, <_t, V \rangle$ for \mathcal{L} can also be regarded as a structure for L_1 . The relation $<_t$ is the extension of the L_1 binary symbol $<$, and V assigns each unary predicate P of L_1 a subset of T , just as required. Moreover we have that:

$$\langle T, <_t, V \rangle \models \phi[t] \text{ iff } \langle T, <_t, V \rangle \models ST(\phi)[t],$$

for all \mathcal{L} wffs ϕ . The occurrence of \models on the right hand side means, of course, the ordinary first order satisfaction relation; and the right hand side $[t]$ is the usual first order notation for describing the assignments of values to (relevant) variables. In this case it means that t_0 has been assigned t .

That the equivalence holds could hardly be more immediate. We could, if we wanted, prove it by induction on $\deg(\phi)$; but as van Benthem observes, in effect all we have done is construe the truth definition of \mathcal{L} rather more prosaically usual: as a translation, not as a 'semantics'. Prosaic but fruitful: it immediately tells us that tense logical validity must be an *re* notion. Admittedly we already knew this (and more) from the filtration argument of Chapter 2, but the result can simply be 'read off' the correspondence. It also tells us that as far as truth in a model is concerned, Löwenheim-Skolem and Compactness theorems must hold. But while truth in a model is interesting, in order to really understand what is involved in standard tense logic we must look at validity on a frame, and considering this leads us from L_1 to L_2 .

Suppose we feed an \mathcal{L} wff ϕ through the standard translation to produce $ST(\phi)$. Now, when we say that $\mathbf{T} \models \phi$ we mean that for all $t \in T$ and all valuations V on \mathbf{T} ,

$\langle T, V \rangle \models \phi[t]$. Can we capture the tense logical validity of an arbitrary \mathcal{L} wffs ϕ by judiciously tinkering with $ST(\phi)$? Obviously to do so we need to universally quantify over t_0 to make the point of evaluation irrelevant, which produces $\forall t_0 ST(\phi)$ — but now we are stuck. In order to express that the particular valuation in force on T is irrelevant we need ~~to~~ to universally quantify over the entities that ‘embody valuations’ in L_1 , namely predicates. In short, we need second order quantification. So, on the spur of the moment we invent L_2 : we regard every predicate in L_1 as a predicate variable, and treat the standard translation as a translation into this new language. Note that models $\langle T, <, V \rangle$ are not structures for L_2 ; just as with L_0 it is the underlying frame that is the structure, and models are L_2 structures coupled with an assignment of values to the L_2 predicate variables, V . It is thus clear that:

$$T \models \phi \text{ iff } T \models \forall P_1 \dots \forall P_n \forall t_0 ST(\phi),$$

where the P_1, \dots, P_n are all and only the L_2 predicate variables in $ST(\phi)$. We have completely abstracted away from the effects of particular valuations, and the choice of point of evaluation, and thus it is clear that the validity of standard tense logical wffs corresponds to the validity of certain Π_1^1 wffs of L_2 . The realisation that this second order perspective on standard tense logic existed was the starting point for much of the important ^{work} in intensional logic of the 1970s. For further discussion of these matters we refer the reader to [5], [13] and references therein. For a detailed examination of the theory this correspondence gives rise to the reader should consult [7] or [6].

Let us now consider what happens when \mathcal{L} is a language with nominals. In fact matters could hardly be simpler: we can treat nominals as corresponding to ordinary L_0 (or L_1 , or L_0 , depending on which target language we are aiming for) *individual variables*. To see this, let’s first impose a little more order on the individual variables of our classical languages. We already have a special designated variable t_0 . Divide the remaining countable infinity of individual variables into two countably infinite sets. One set, whose elements we will write as x, y, z and so on, will be used solely as the ‘bookkeeping’ binding variables in the translation clauses for wffs of the form $F\phi$ and $P\phi$. The other set, whose elements we will write as i, j, k and so on, will be used solely

to represent the nominals. Given this convention, to translate a language with nominals into our classical languages we need merely add the clause:

$$ST(i) = (i = t_0),$$

for all nominals i . Note that when we translate an \mathcal{L} wff ϕ containing m nominals we obtain a classical formula with $m + 1$ free individual variables: t_0 , and a free variable for each nominal.

The needed equivalence still holds — once more we are merely treating our truth definition prosaically. If we are translating into L_2 we can still write the equivalence as:

$$\langle T, <_t, V \rangle \models \phi[t] \text{ iff } \langle T, <_t, V \rangle \models ST(\phi)[t],$$

without too much obscurity. Note that the V on the right hand side is an assignment of values to both predicate variables and those individual variables that correspond to nominals. (Strictly, first order assignments of values to individual variables assign elements of the domain of quantification, not singleton sets of such elements as V does — but the intent is clear.) On the other hand, if we are translating into L_1 this notation is rather sloppy as V now embodies two types of information kept distinct in classical languages: the denotations of the predicate constants, which are part of the structure; and the assignment of values to variables, which are not. In this it is better to split the information in V into two functions. Let V be written as $V^P \cup V^N$, where the first component contains all the assignments to variables, and the second all assignments to nominals.¹¹ We can then write our equivalence more naturally as

$$\langle T, <_t, V \rangle \models \phi[t] \text{ iff } \langle T, <_t, V^P \rangle \models ST(\phi)[t, t_1, \dots, t_m],$$

where $t_1 \in V^N(i_1), \dots, t_m \in V^N(i_m)$, and the i_1, \dots, i_m are all and only the nominals occurring in ϕ .

¹¹That is, $V^P = V \setminus \{\langle i, S \rangle : i \in NOM_{\mathcal{L}} \text{ and } S \subseteq T\}$ and $V^N = V \setminus \{\langle p, S \rangle : p \in VAR_{\mathcal{L}} \text{ and } S \subseteq T\}$.

Once again, from the translation into L_1 we can derive all sorts of information concerning truth in an NTL model; but the main interest comes from considering validity. Firstly note that if ϕ is a purely nominal wff then $ST(\phi)$ is an L_0 wff: it contains no unary predicates or predicate variables, simply occurrences of $<$, $=$, individual variables and logical connectives. Now, to abstract away from the point of evaluation we universally quantify over t_0 as before; and to abstract away from the effects of any particular valuation we need to universally quantify over the entities^{that} embody valuations in $ST(\phi)$ — but these are now simply the individual variables that correspond to ϕ 's nominals. That is we can express the fact that ϕ is valid by means of *first order* quantification of $ST(\phi)$: for all frames T and all purely nominal wffs ϕ we have:

$$T \models \phi \text{ iff } T \models \forall i_1 \dots \forall i_m \forall t_0 ST(\phi),$$

where the i_1, \dots, i_m are all and only the nominals in ϕ .

This immediately gives us an interesting crop of results concerning *frames* for purely nominal languages \mathcal{L} , or for sets of purely nominal sentences drawn from mixed languages. Firstly, purely nominal frame consequence is an *re* notion, for by the previous correspondence:

$$\Sigma \models_f \phi \text{ iff } ST(\Sigma) \models ST(\phi),$$

where the \models on the right hand side is the ordinary first order consequence relation and $ST(\Sigma) = \{ST(\sigma) : \sigma \in \Sigma\}$. As the first order consequence relation is *re*, so \models_f must be also. In similar fashion we gain Löwenheim-Skolem and Compactness results on frames for purely nominal sets of sentences. Validity on frames is a much simpler notion for nominals than it is for variables; the addition of nominals has filled a useful expressive gap, but it hasn't plunged us into baroque new realms of complexity.

Before going any further, some examples. F_i defines the universal relation, and indeed $ST(F_i)$ is $\forall i \forall t_0 \exists x (t_0 < x \wedge x = i)$, or more simply, $\forall i \forall t_0 (t_0 < i)$, universality. We know that $i \rightarrow \neg F_i$ defines irreflexivity, and if we evaluate $ST(i \rightarrow \neg F_i)$ we obtain

$$\forall i \forall t_0 (i = t_0 \rightarrow \neg \exists x (t_0 < x \wedge i = x)),$$

which simplifies to $\forall t_0 \forall x (t_0 = x \rightarrow t_0 \not< x)$, a somewhat eccentric way of expressing irreflexivity in L_0 . More interestingly, consider Löb's formula, $G(Gp \rightarrow p) \rightarrow Gp$. This

does not express an L_0 condition, nonetheless its nominalisation $G(Gi \rightarrow i) \rightarrow Gi$ must. Writing out the standard translation we obtain:

$$\forall i \forall t_0 (\forall x (t_0 < x \rightarrow (\forall y (x < y \rightarrow i = y) \rightarrow i = x) \rightarrow \forall z (t_0 < z \rightarrow z = i)).$$

What happens if \mathcal{L} is a mixed language of NTL? If ϕ is a wff of such a language then it is clear by what has gone before that

$$\mathbf{T} \models \phi \text{ iff } \mathbf{T} \models \forall P_1 \dots \forall P_n \forall i_1 \dots \forall i_m \forall t_0 ST(\phi).$$

That is validity of mixed wffs, like the validity of purely Priorean wffs, corresponds to the validity of certain Π_1^1 wffs of L_2 .

The existence of the standard translation for languages of NTL raises a number of interesting questions. Let's first consider mixed language. Can we define any L_2 conditions in mixed languages that are not definable in purely Priorean languages? We already know the answer to be yes — ϕ^Z defines the integers and ϕ^N the natural numbers, and neither frame is definable in purely Priorean languages. Beyond this simple observation matters are unclear. One pressing and difficult question is the following: precisely what are the L_0 expressible classes of frames definable in mixed languages? The obvious way to attack this problem would be to adapt the algebraic methods of Goldblatt and Thomason [37]; however there is a difficulty. As NTL validity is not preserved under disjoint unions, this has the algebraic effect of blocking steps involving products of algebras. The necessary and sufficient condition we have for NTL validity to be so preserved may allow some results to be obtained, but in general I think that answering this question by any method is going to be hard.

Let's turn to purely nominal languages. We know that these can only define L_0 conditions, and we also know that some of these conditions — such as irreflexivity — are not definable in any purely Priorean language. Now, Priorean languages, because they can define higher order conditions, can define conditions no purely nominal language can — but can they define any first order conditions that purely nominal languages cannot? So far we have seen no counterexamples; could it be that as far as L_0 conditions are concerned, purely nominal languages are stronger than purely Priorean ones? The

answer is no, but some work is required to see this. Both the counterexample that follows and the idea underlying the proof are due to Johan van Benthem.

The counterexample is ‘transitivity plus atomicity’. An atomic frame is one in which every point x precedes an ‘atom’ y that is its own only successor:

$$\forall x \exists y (x < y \wedge \forall z (y < z \rightarrow z = y));$$

and the class of all frames that are both transitive and atomic are definable in purely Priorean languages by the conjunction of 4 with $GFp \rightarrow FGP$, McKinsey’s axiom.¹² However no purely nominal wff can define this condition. The essence of the argument that follows is this: any such formula ϕ which putatively defines this condition can be falsified on N , as N contains no atoms. Because of a certain ‘stability property’ we will demonstrate, it is possible to by means of filtration to turn this falsifying model into a transitive and atomic falsifying model, showing that no such wff can define the desired class.

First the ‘stability lemma’. Its intuitive content is this: given any purely nominal wff ϕ and a valuation on N , by moving sufficiently far to the right along N we reach a point where the truth values of ϕ and all its subformulas settle down to some fixed values. This is because eventually we reach a point where all nominals in ϕ denote points in the past — we’ve overshot their denotations. The only tricky part in establishing this is driving through the clause for formulas of the form $P\phi$, as such formulas can look back at points before things settled down; this motivates use of the td measure in the following lemma:

Lemma 3.3.1 *Let Σ be a set of sentences closed under subformulas such that the only atoms in Σ are a finite collection of nominals. Let V be a valuation on N , and let $l - 1$*

¹²The proof that this suffices requires use of the axiom of choice. In passing, McKinsey’s axiom does *not* define atomicity — it only defines it on the transitive frames — and in fact does not correspond to any L_0 condition. For further details the reader should consult [7, 203] or [6, page 111].

be the largest natural number that V assigns to some nominal in Σ . That is, $l - 1 = \max \bigcup_{i \in \Sigma} V(i)$. Then for all wffs $\sigma \in \Sigma$, for all $n > l + td(\sigma)$, $V(\sigma, n) = V(\sigma, l + td(\sigma))$.

Proof:

Induction on $td(\sigma)$. Suppose $td(\sigma) = 0$. Then as σ is a purely nominal sentence it is either a nominal or a boolean combination of nominals, all of which are in Σ . As all nominals in Σ are false from l onwards, the result is clear by induction on $deg(\sigma)$.

Assume the result holds for all $\sigma' \in \Sigma$ such that $td(\sigma') < m$, where $m > 0$. Suppose $td(\sigma) = m$. We want to show that for all $n > l + m$, $V(\sigma, n) = V(\sigma, l + m)$.

Suppose σ has the form $P\psi$. Clearly $V(P\psi, l + m) = 1$ implies $V(P\psi, n) = 1$ for all $n > l + m$. So suppose that $V(P\psi, l + m) = -1$. Then for all $h < l + m$ we have that $V(\psi, h) = -1$, which in particular means that $V(P\psi, l + (m - 1)) = -1$. As $td(\psi) = m - 1$, by the inductive hypothesis we have that for all $V(\psi, n) = -1$ for all $n > l + m$, which means that ψ is false everywhere on \mathbb{N} . Thus trivially $V(P\psi, n) = -1$ for all $n > l + m$, as required. Alternatively, if we assume that σ has the form $F\psi$ a similarly styled argument also gives the required result.

The only other possibility is that σ is a boolean combination of elements $\sigma'_1, \dots, \sigma'_k$ of Σ such that $td(\sigma'_j) < m$ or $td(\sigma'_j) = m$ and σ'_j has the form $P\psi$ or $F\psi$. But our work so far tells us that for all such σ'_j ($1 \leq j \leq k$), $V(\sigma'_j, n) = V(\sigma'_j, l + td(\sigma'_j))$, for all $n > l + td(\sigma'_j)$, and as σ is a boolean combination of such forms an easy inductive argument shows that $V(\sigma, l + m) = V(\sigma, n)$ for all $n > l + m$. \square

Lemma 3.3.2 *If a purely nominal wff ϕ is falsifiable on \mathbb{N} , then ϕ is falsifiable in a (finite) transitive and atomic model.*

Proof:

Let ϕ be a purely nominal wff such that for some valuation V' and point $k \in N$, $\langle \mathbb{N}, V' \rangle \not\models \phi[k]$. Let Σ^- be the smallest set of sentences containing ϕ that is closed under subformulas, and let $l - 1$ be the largest natural number that V' assigns to any nominal

in Σ^- . Let $td(\phi) = c$. As for all $\sigma \in \Sigma^-$ $td(\sigma) \leq c$ we know by the previous lemma that for all $n > l + c$ the truth values in $\langle N, V' \rangle$ are stable.

Let i be any nominal not occurring in Σ^- , and V be the valuation that is just like V' save possibly that $V(i) = \{(l + c) - 1\}$. As V and V' agree on the values of all atoms in Σ^- , by the Agreement Lemma we have that for all $\sigma \in \Sigma^-$, for all $n > l + c$, $V(\sigma, n) = V(\sigma, l + c)$, and moreover $\langle N, V \rangle$ also falsifies ϕ at k . Let $\Sigma = \Sigma^- \cup \{Fi, i\}$. Clearly for all $\sigma \in \Sigma$, for all $n > l + c$, $V(\sigma, n) = V(\sigma, l + c)$. Note that Σ is a finite set of sentences closed under subformulas.

Prior to filtrate $\langle N, V \rangle$ through Σ to form M^f . M^f falsifies ϕ at $E(k)$ by the Filtration Theorem; M^f is transitive because we took a Priorean filtration; and M^f is finite because Σ was finite. If we can show that M^f is atomic we are through.

We know that in the model $\langle N, V \rangle$, the truth values of all wffs $\sigma \in \Sigma$ are stable from $l + c$ onwards, but this means that all $n \geq l + c$ are in the same equivalence class. Call the element of M^f of which they are all a member $E(l + c)$. We now show that $E(l + c)$ is an atom that all other elements of M^f precede.

In any filtration whatsoever, if $t < t'$ in the original model then $E(t) <_f E(t')$ in the filtration. Hence any other element $E(t) \in M^f$ must precede $E(l + c)$. Moreover it follows from the definition of $<_f$ coupled with the stability of the truth values of the wffs in Σ that $E(l + c) <_f E(l + c)$. Thus the only thing that could prevent $E(l + c)$ from being the desired atom would be if $E(l + c)$ preceded some distinct $E(t) \in M^f$. We now show that this is impossible.

Suppose $E(t) \neq E(l + c)$ and $E(l + c) <_f E(t)$. We need merely note that no $h \in N$ such that $h < l + c$ can be in $E(t)$. For if $h < l + c$ then $\langle N, V \rangle \models Fi[h]$, and if $h = (l + c) - 1$ then $\langle N, V \rangle \models i[h]$, and as both Fi and i are in Σ we have by the Filtration Theorem that $M^f \models Fi[E(t)]$ or $M^f \models i[E(t)]$, which by the definition of $<_f$ would mean that $M^f \models Fi[E(l + c)]$. But another appeal to the Filtration Theorem shows that this in turn would mean that $\langle N, V \rangle \models Fi[l + c]$, which is impossible as $V(i) = \{(l + c) - 1\}$. In short, any such $E(t)$ would be empty, so $E(l + c)$ precedes no point save itself, is the required atom, and we are through. \square

Thus transitivity plus atomicity is a first order condition definable in a purely Priorean language that is not purely nominal definable: as far as expressing L_0 conditions is concerned the two sorts overlap.

What L_0 conditions can be defined using purely nominal wffs? At present I don't know, but am investigating the question by looking at the forms of the L_0 wffs produced by the standard translation; this question promises to be easier to answer than the analogous question for mixed languages. Finally, we know the class of first order conditions that are purely nominal definable are re; are they also recursive? With this open question I'll conclude the discussion of the first order defining power of nominals; clearly much interesting work remains to be done.

3.3.2 D Logic

The D operator is a relatively new addition to intensional languages. Ron Koymans invented it in the course of investigating applications of temporal logics in computer science, and used it to help specify message passing systems [57]. Independent work by Valentin Goranko [42] at around the same time also considered the operator, and more recently Maarten de Rijke [89] has extensively investigated its logical properties.

Like F and P , the D operator is a unary propositional operator, and $D\phi$ reads ' ϕ is true at a *different* point'. Let us now consider what happens when the D operator is added to standard languages of tense logic. Given a purely Priorean language \mathcal{L} add a distinct new symbol D ; augment the definition of the well formed formulas by stipulating that if ϕ is a wff, $D\phi$ is also; and interpret these new wffs in any model \mathbf{M} via the new semantic clause:

$$\mathbf{M} \models D\phi[t] \text{ iff } \exists t' \neq t \text{ such that } \mathbf{M} \models \phi[t'].$$

As Koymans notes [57, Chapter 4], the expressive power of the D augmented language greatly outstrips that of the purely Priorean original.

Firstly, a variety of powerful new operators are definable in terms of D . The most obvious is D 's dual, \overline{D} . $\overline{D}\phi$ has the reading 'At no different point does $\neg\phi$ hold', or

more simply, ‘ ϕ is true everywhere else’. Defining $E\phi$ to be $\phi \vee D\phi$ yields an operator reading ‘there *exists* a point at which ϕ is true’; and defining $A\phi$ to be $\phi \wedge \overline{D}\phi$ we obtain a universal (S5) operator — $A\phi$ reads ‘ ϕ holds at *all* points’. Finally we can define a uniqueness operator U by defining $U\phi$ to be $E(\phi \wedge \neg D\phi)$ — ϕ is true at exactly one point. As we will see, this operator allows D logic to simulate nominals.

Secondly, new classes of frames not definable in the purely Priorean base language become definable with the aid of D . The irreflexive frames can be defined by $F\phi \rightarrow D\phi$, the trichotomous frames by $D\phi \rightarrow (P\phi \vee F\phi)$, and the discrete frames by

$$(P(p \wedge \neg Dp) \rightarrow E(Pp \wedge \neg PPp)) \wedge (F(p \wedge \neg Dp) \rightarrow E(Fp \wedge \neg FFp)).$$

In fact a little experience working with D logic makes it clear that the difference operator is a very powerful tool. This is testified to by the fact that not only is D validity not preserved under either the formation of p-morphic images or disjoint unions — this much is clear already by the definability of irreflexivity and trichotomy — but, as we will shortly see, D validity is not even transmitted to generated subframes. In fact, of the four preservation results already discussed, only the result for ultrafilter extensions still obtains in D logic; this was shown by Maarten de Rijke [89, pages 5–7]. Given this, it is natural to suspect that our D augmented language is more expressive than any mixed language of NTL. This is so and may be seen as follows.

Firstly, using D we can define conditions not definable in any language with nominals. We have already noted that because NTL validity is inherited by generated subframes we cannot define the class of frames containing an isolated irreflexive point. But this class is definable with the help of D , by $E(H\perp \wedge G\perp)$; an example which also shows that D validity is not transmitted to generated subframes. As a second example, we know that the only cardinality definable in NTL is 1 — recall that this was also due to the generated subframe result for NTL. However, as Ron Koymans proved, given any finite number n , the class of all frames of cardinality n is definable in D logic. For further details see [57, page 50].

The converse does not hold: there are no conditions definable in mixed languages of NTL which D logic cannot define, as the following translation of NTL into D logic noted

by Yde Venema makes clear. Let $\phi(i_1, \dots, i_n)$ be any wff of a mixed language of NTL, where the i_m ($1 \leq m \leq n$) are all and only the nominals in ϕ . Let p_{i_1}, \dots, p_{i_n} be n distinct variables not occurring in ϕ . Then for any frame \mathbf{T} we have that:

$$\mathbf{T} \models \phi(i_1, \dots, i_n) \text{ iff } \mathbf{T} \models U p_{i_1} \wedge \dots \wedge U p_{i_n} \rightarrow [p_{i_1}/i_1, \dots, p_{i_n}/i_n]\phi,$$

as a straightforward argument shows. Because of the U operator, D logic in effect already contains nominals.

This correspondence gives us an upper bound on the expressive power of NTL. In fact the ‘ D umbrella’ will stay in place even when we introduce interval nominals in Chapter 6.

3.4 Further directions

Some useful things have been achieved in this chapter. Firstly, we have seen that languages with nominals can talk about important classes of frames that purely Priorean languages cannot. Secondly, our examination of the four classic tense logical preservation results has given us some insight into how the functioning of nominals differs from that of variables; and in addition, we have gained two important model equivalence results — the generated submodel and p-morphic model results — that will play a crucial role in establishing the completeness theorems of later chapters. Thirdly, we have established the basic correspondences between NTL, three classical languages, and D logic. But though we have the beginnings of a useful model theory, much important work remains to be done. The matters concerning correspondence that should be investigated have already been mentioned, and I will not discuss them further here; rather, I will close this chapter by considering another important topic that should be examined: isolating purely nominal preservation results.

Our examination of preservation results so far has been one way. We have two preservation results which hold for both nominals and variables, and two preservation results which hold for variables only. What is missing are preservation results which

hold for nominals and fail for variables. I will now sketch some ideas involving paths that seem to hold some promise of leading to such results.

In the previous chapter I stated that paths and ‘path equations’ were an important guiding intuition when working with nominals.¹³ Actually that remark was rather circumspect; I believe they may hold the key to the sort of preservation results we require, and that we should examine what operations on paths are permitted — in particular, when can we identify points on paths, and when can we pull points apart. To make matters concrete, let’s consider the class of Church Rosser frames, that is, the class of all frames such that

$$\forall t(t < t' \wedge t < t'' \rightarrow \exists c(t' < c \wedge t'' < c)).$$

The Church Rosser frames can be defined by the purely Priorean formula $FGp \rightarrow Gfp$. Note how this works: the p in the antecedent ‘labels’ a set of points, namely all the successors of some successor of the point of evaluation. The consequent then asserts that from any successor of the point of evaluation some such labelled point can be reached. So, speaking rather loosely, we could say that purely Priorean languages define this condition by ‘referring to’ the potential points of convergence — and the only class of frames in which this reference is bound to succeed are the Church Rosser frames. Note that the defining wff uses only forward looking operators.

¹³Some readers may find this a truism. Surely we can think of any constraint on frame orderings as a sort of path equation; and do not purely Priorean languages define path equations as well? After all, $FFp \rightarrow Fp$ and $FFi \rightarrow Fi$ both define the same class of frames — why should we think of the latter and not the former as a path equation? All of this is perfectly true; but the point is that the two sorts differ in their defining capacities although there is some overlap, and for a number of reasons the way the unique labelling capacity of nominals interacts with the tense operators often seems appropriately described as a path equation. However the terminology is not important: what is at stake is the substantive difference between the sorts. This path equation *metaphor*, if you will, is simply a way of thinking about a number of difficulties that at present I find helpful.

The nominalisation $FGi \rightarrow GFi$ of this wff does not define the Church Rosser frames.

¹⁴ A counterexample is provided by the frame I call the V naturals, $Vee(N)$. This consists of two copies of N with the two zeroes identified. (That is, it is a V shaped frame consisting of two N -alikes springing disjointly from a common source.) Clearly $FGi \rightarrow GFi$ is valid on $Vee(N)$ as its antecedent must always be false; however $Vee(N)$ is not Church Rosser because of the common zero.

A little experimentation will rapidly persuade the reader that the Church Rosser frames cannot be defined by a purely nominal wff if only forward looking operators are used. This intuition is correct, as a simple variant of the filtration argument used to show that transitivity plus atomicity was not purely nominal definable shows. First we need a stability lemma for the V naturals:

Lemma 3.4.1 *Let Σ be a set of sentences closed under subformulas such that the only atoms in Σ are a finite collection of nominals, and such that no wff $\sigma \in \Sigma$ contains occurrences of P or H . Let V be any valuation on $Vee(N)$, and let $l - 1$ denote the largest number on either branch of the frame that V assigns to any nominal in Σ . Then, on both branches of the frame, we have that for all $n > l$, $V(\sigma, n) = V(\sigma, l)$, for all $\sigma \in \Sigma$.*

Proof:

Essentially the same as the proof of the earlier stability lemma, but simpler: because there are no wffs containing P or H an induction on $deg(\sigma)$ is all that is required. \square

We now show using filtrations that any forward looking purely nominal formula falsifiable on the V naturals is falsifiable on a Church Rosser frame.

Lemma 3.4.2 *If a purely nominal wff ϕ containing no occurrences of P or H operators is falsifiable on $Vee(N)$, then ϕ is falsifiable on a (finite transitive) Church Rosser model.*

¹⁴ $FGi \rightarrow GFi$ defines the condition $\forall i \forall t_0 \forall x \forall y (t_0 < x \wedge t_0 < y \wedge \forall z (x < z \rightarrow z = i) \rightarrow y < i)$, as the standard translation shows.

Proof:

Given such a wff ϕ and a falsifying valuation V for it on $Vee(\mathbf{N})$, let Σ be the smallest subformulae closed set of sentences containing ϕ , and Priorean filtrate $\langle Vee(\mathbf{N}), V \rangle$ through Σ forming the finite transitive model \mathbf{M}^f . This model also falsifies ϕ by the Filtration Theorem.

In fact \mathbf{M}^f is also Church Rosser. By the previous lemma we have that all points $n \geq l$ in \mathbf{N} are identified in \mathbf{M}^f . Call this point $E(l)$. We then have that $E(t) \neq E(l)$ implies $E(t) <_f E(l)$. This follows from the morphic property of filtrations: for any filtration, $t < t'$ in the original model implies $E(t) <_f E(t')$ in the filtration. Clearly any point in an equivalence class distinct from $E(l)$ must precede some point in $E(l)$, and this is all we need. It is also easy to show, using the stability lemma and the definition of $<_f$, that $E(l) <_f E(l)$. But this means that $E(l)$ succeeds every point in \mathbf{M}^f , thus it will serve as a convergence point for any two points in the frame, and thus \mathbf{M}^f is Church Rosser. \square

Note that this also a proof that right directedness is not purely nominal definable in NML: the V naturals are not right directed, but the filtration is as $E(l)$ (being maximal) right directs the entire frame.

By making use of both forward and backward looking operators we can define the Church Rosser frames: $F i \wedge F j \rightarrow F(i \wedge F P j)$ achieves this. Note that neither the i nor the j refers to the convergence point; rather they refer to the two successors of the point of evaluation, and the convergence point is implicitly picked out by the $F P j$ in the consequent which ‘curls around’ it. Essentially the formula works by labelling the start and finish points and asserting that an appropriate path between^{them} exists, which is very different from the way that $F G p \rightarrow G F p$ worked.

Obviously the filtration plus stability argument used here and in the discussion of transitivity plus atomicity is rather specific, but I think some suggestive ideas can be abstracted from it. Firstly, going from $Vee(\mathbf{N})$ to \mathbf{M}^f (or from \mathbf{N} to \mathbf{M}^f) we have collapsed paths. Looking at the process in reverse we see a way of expanding or ‘unwinding’ paths. Now this was the sort of thing we were looking for, but given that the method of collapse

was merely filtration, and, as we will become plain in Chapter 5, that the expansion involved was essentially a form of bulldozing, this may seem rather unexciting. What I think is interesting is the way that the transformations were ‘local’, or ‘structurally specific’. For example, looking at the transformation from the big models to M' we see that path collapsing was permitted because the underlying frame geometry — in this case, essentially, the fact that they were right unbounded SPOs — enabled us always to find regions where nominals were distributed in a fashion that was amenable to analysis. This suggests to me that it may be fruitful to look for such local, structurally dependent preservation results. Further ideas that bear on this issue are mentioned in Chapter 6.

This concludes our examination of the model theory of NTL; now we consider how to control nominals proof theoretically.

Chapter 4

The Minimal Logic

In this chapter we axiomatise the minimal logic in languages of NTL. The first section outlines the background ideas needed, and after this we present an axiomatisation which consists of the usual axiomatisation K_t of minimal tense logic augmented by a single infinite schema called NOM. We first show a negative result: we can't use the usual canonical Henkin model argument unmodified to establish completeness, and then prove completeness by adapting a method due to David Makinson. An examination of the proof shows that it worked because the construction yielded a connected frame; in the course of this discussion we uncover three more axiomatic bases for the minimal tense logic — the schema SWEEP, and two weakened schema, NOM_w and SWEEP_w — and present a shorter completeness proof using generated subframes of the canonical frame. After discussing the minimal tense logic we turn to the minimal nominal modal logic. That either the modal versions of the NOM or the SWEEP schema can be used to axiomatise this logic is apparent, but we show that neither of the weakened schema will suffice, and briefly discuss the impact the addition of nominals has on the Henkin frame of the minimal modal logic. We conclude the chapter by examining the axiomatisation of minimal nominal modal logic due to Gargov, Passy and Tinchev [34].

4.1 Notions and notations

Let \mathcal{L} be a language of NTL. By $PC_{\mathcal{L}}$ is meant the set of all \mathcal{L} -instances of tautologies of propositional calculus. If $\phi \in PC_{\mathcal{L}}$ then ϕ is called an \mathcal{L} -tautology. If $L \subseteq WFF_{\mathcal{L}}$ then we say that L is an \mathcal{L} -logic iff $PC_{\mathcal{L}} \subseteq L$ and L is closed under modus ponens. For the sake of simplicity we usually assume \mathcal{L} has been fixed and talk simply of L being a logic. $PC_{\mathcal{L}}$ and $WFF_{\mathcal{L}}$ are both logics and for any logic L (in language \mathcal{L}), $PC_{\mathcal{L}} \subseteq L \subseteq WFF_{\mathcal{L}}$. A *tense logic* in language \mathcal{L} is a logic that contains all \mathcal{L} -instances of $G(\phi \rightarrow \psi) \rightarrow (G\phi \rightarrow G\psi)$, $H(\phi \rightarrow \psi) \rightarrow (H\phi \rightarrow H\psi)$, $\phi \rightarrow GP\phi$, and $\phi \rightarrow HF\phi$. and is closed under *temporal generalisation*; that is, $\phi \in L$ implies $G\phi$ and $H\phi \in L$. Note that if \mathcal{M} is a class of models and \mathcal{T} a class of frames then both $Th(\mathcal{M})$ and $Th(\mathcal{T})$ are tense logics.

If L is an \mathcal{L} -logic and $\Sigma \subseteq WFF_{\mathcal{L}}$ and $\phi \in WFF_{\mathcal{L}}$, then we say that ϕ is *L -derivable from assumptions Σ* , written $\Sigma \vdash_L \phi$, iff there are $\sigma_1, \dots, \sigma_n \in \Sigma$ such that $\sigma_1 \wedge \dots \wedge \sigma_n \rightarrow \phi \in L$. While Σ may be infinite, if $\Sigma \vdash_L \phi$ then there exists a *finite* $\Sigma' \subseteq \Sigma$ such that $\Sigma' \vdash_L \phi$, namely the set consisting of precisely $\sigma_1, \dots, \sigma_n$. In the special case where $\Sigma = \emptyset$ we mean that $\emptyset \vdash_L \phi$ iff $\phi \in L$ — in which case we say that ϕ is *L -derivable* and write $\vdash_L \phi$. If ϕ is not L -derivable from assumptions Σ we write $\Sigma \not\vdash_L \phi$. Note that for any logic L both the deduction theorem and its converse hold. That is, $\Sigma \cup \{\phi\} \vdash_L \psi$ iff $\Sigma \vdash_L \phi \rightarrow \psi$. $\Sigma \subseteq WFF_{\mathcal{L}}$ is *L -inconsistent* iff for some $\phi \in WFF_{\mathcal{L}}$ both ϕ and $\neg\phi$ are L -derivable from assumptions Σ . Σ is *L -consistent* iff Σ is not L -inconsistent. Σ is *L -maximal consistent* — an *L -MCS* — iff Σ is L -consistent, and for all \mathcal{L} -wffs ϕ either ϕ or $\neg\phi \in \Sigma$. Note that this can hold iff Σ is not contained in any other consistent Σ' . The following properties hold of any L -MCS Σ : $\phi, \psi \in \Sigma$ iff $\phi \wedge \psi \in \Sigma$; $\phi \in \Sigma$ iff $\neg\phi \notin \Sigma$; $\phi, \phi \rightarrow \psi \in \Sigma$ implies $\psi \in \Sigma$; and $\vdash_L \phi$ implies $\phi \in \Sigma$. The proofs are straightforward. More importantly, we can make all the L -MCS we require, as Lindenbaum's Lemma holds.

Lemma 4.1.1 (Lindenbaum's Lemma) *Every L -consistent set of sentences Σ can be extended to an L -MCS Σ^∞ .*

Proof:

As usual. Index $WFF_{\mathcal{L}}$ by some ordinal α . Define Σ_0 to be Σ . For all successor ordinals $\beta \leq \alpha$ define $\Sigma_{\beta+1}$ to be $\Sigma_\beta \cup \{\phi_\beta\}$ if $\Sigma_\beta \vdash_L \phi_\beta$, otherwise set $\Sigma_{\beta+1} = \Sigma_\beta$. For all limit ordinals $\lambda \leq \alpha$ let Σ_λ be $\bigcup_{\gamma < \lambda} \Sigma_\gamma$. It is easy to show, using facts of propositional calculus, that L -consistency is upward preserved from Σ to Σ_α . But Σ_α is clearly an L -MCS, and $\Sigma \subseteq \Sigma_\alpha$. \square

Note that once an enumeration of the wffs of \mathcal{L} has been fixed, the above construction yields a *unique* maximal consistent extension of any Σ . We assume that each language \mathcal{L} comes equipped with a standard enumeration of its wffs, and from now on given an L -consistent set of sentences Σ in language \mathcal{L} , by Σ^∞ is meant the L -MCS formed by the the above construction with respect to the standard enumeration for \mathcal{L} .

Definition 4.1.1 (Soundness and Weak Completeness) *Let \mathcal{T} be a class of frames and L a logic. L is sound with respect to \mathcal{T} iff for all wffs ϕ , $\vdash_L \phi$ implies $\mathcal{T} \models \phi$; and L is weakly complete with respect to \mathcal{T} iff for all wffs ϕ , $\mathcal{T} \models \phi$ implies $\vdash_L \phi$. If L is both sound and weakly complete with respect to \mathcal{T} then L is characterised by \mathcal{T} .* \square

Modal completeness theory has many examples of logics L characterised by more than one class of frames. A simple example is provided by S4. The classic result is that S4 is complete with respect to the preorders; but as Segerberg noted [95], this can be sharpened to a result stating that S4 is complete with respect to the partial orders. A subtheme of the next chapter is the way that certain of these classic 'sharpening' results are lost in languages with nominals.

The previous definition linked the semantical concept of \mathcal{T} -validity with the syntactical notion of L -derivability, and under one conception of what logic is — logic as the study of valid sentences — our task is precisely to prove such characterisation results. But logic is also the study of valid patterns of inference, or correct reasoning. What could

be meant by ‘correct temporal reasoning’ — or at least, correct temporal reasoning for native NTL speakers?

One sensible attempt to provide such a concept is provided by the notion of local model consequence defined in Chapter 1 by $\Sigma \models_m \phi$ iff $\mathbf{M} \models \Sigma[t]$ implies $\mathbf{M} \models \phi[t]$, for all models \mathbf{M} , and all points t of \mathbf{M} . This relation tells us how ‘facts of the matter’ are mutually constrained; it takes us from states of affairs to their concomitants. However in general we only wish to reason within models constructed on a small subclass of the possible frames, namely those frames that reflect our views of temporal ontology. We should parameterise the definition of model consequence by classes of frames, as follows:

Definition 4.1.2 (Local \mathcal{T} -based model consequence) *Let \mathcal{T} be a class of frames, Σ a set of wffs and ϕ a wff. Then $\Sigma \models_m^{\mathcal{T}} \phi$ iff $\langle \mathbf{T}, V \rangle \models \Sigma[t]$ implies $\langle \mathbf{T}, V \rangle \models \phi[t]$ for all $\mathbf{T} \in \mathcal{T}$, $V \in \text{Val}(\mathbf{T})$ and $t \in \mathbf{T}$. \square*

That is, $\Sigma \models_m^{\mathcal{T}} \phi$ iff any model constructed on a frame in \mathcal{T} making Σ true at t , makes ϕ true at t also. Note that the previous notion of local model consequence is the special case of \mathcal{T} -model consequence with $\mathcal{T} = \mathcal{U}$, the class of all frames. That is, $\Sigma \models_m \phi$ iff $\Sigma \models_m^{\mathcal{U}} \phi$.

Definition 4.1.3 (Strong Completeness) *Let \mathcal{T} be a class of frames and L a logic. Then L is said to be strongly complete with respect to \mathcal{T} iff $\Sigma \models_m^{\mathcal{T}} \phi$ implies $\Sigma \vdash_L \phi$ for all sets of wff Σ , and all wffs ϕ . \square*

In short, a logic L is strongly complete with respect to a class of frames \mathcal{T} if the syntactic notion of L -derivability from assumptions captures the semantic notion of local \mathcal{T} -based model consequence. If a logic L is strongly complete with respect to \mathcal{T} then it is weakly complete with respect to \mathcal{T} as well, however the converse does not hold as certain logics are not compact. (An example is provided by N ; although any finite subset of $\{FG\neg p, Fp, F^2p, F^3p, \dots\}$ is satisfiable on N , the entire set is not.) The following lemma is standard:

Lemma 4.1.2 (Strong Completeness Lemma) *Let L be a logic and \mathcal{T} a class of frames. L is strongly complete with respect to \mathcal{T} iff for every L -consistent set of sentences Σ there is a model $\langle \mathbf{T}, V \rangle$ such that $\mathbf{T} \in \mathcal{T}$, V is a valuation on \mathbf{T} and $\langle \mathbf{T}, V \rangle \models \Sigma$. (That is, L is strongly complete with respect to \mathcal{T} iff any L -consistent set of sentences is satisfiable in a \mathcal{T} -based model.)* \square

By an axiomatisation in a language \mathcal{L} is meant the following. We indicate that a certain subset A of $WFF_{\mathcal{L}}$ are *axioms*. If \mathcal{L} is a recursive language, A will always be a recursive subset of $WFF_{\mathcal{L}}$. Among our axioms will always be all \mathcal{L} instances of:

- PC1** $\phi \rightarrow (\psi \rightarrow \phi)$
- PC2** $(\phi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \theta))$
- PC3** $(\neg\psi \rightarrow \neg\phi) \rightarrow (\phi \rightarrow \psi)$
- TL1** $G(\phi \rightarrow \psi) \rightarrow (G\phi \rightarrow G\psi)$
- TL2** $H(\phi \rightarrow \psi) \rightarrow (H\phi \rightarrow H\psi)$
- TL3** $\phi \rightarrow GP\phi$
- TL4** $\phi \rightarrow HF\phi$

The following three functions, called *rules of inference*, will always be available:

- MP $_{\mathcal{L}}$** $\{ \langle \phi, \phi \rightarrow \psi, \psi \rangle : \phi, \psi \in WFF_{\mathcal{L}} \}$
- HGen $_{\mathcal{L}}$** $\{ \langle \phi, H\phi \rangle : \phi \in WFF_{\mathcal{L}} \}$
- GGen $_{\mathcal{L}}$** $\{ \langle \phi, G\phi \rangle : \phi \in WFF_{\mathcal{L}} \}$

and are referred to as the rules of Modus Ponens, H-Generalisation and G-Generalisation in \mathcal{L} respectively. The last two rules are referred to collectively as the rules of temporal generalisation in \mathcal{L} . The collection of axioms A and the three rules of inference in any language \mathcal{L} will be called an *axiomatisation*. The axiomatisation consisting of precisely PC1-PC3, TL1-TL4, modus ponens and temporal generalisation is traditionally called K_t ; it yields all and only the validities of standard tense logic.

Let A be an axiomatisation. An *A-proof* is a finite sequence of \mathcal{L} -wffs such that each wff ϕ in the sequence is either an axiom, the result of applying $MP_{\mathcal{L}}$ to two earlier items in the sequence, or the result of applying one of the rules of temporal generalisation to

an earlier item in the sequence. An \mathcal{L} -wff ψ is a *theorem* of A iff there is an A -proof with ψ as the final item.

Let Σ be a set of \mathcal{L} -wffs. By an *A-proof from assumptions* Σ is meant a finite sequence such that every wff ϕ in the sequence is either an axiom, an element of Σ , the result of an application of $MP_{\mathcal{L}}$ to two earlier items, or the result of applying one of the rules of temporal generalisation to an A -theorem. This last restriction is crucial: it is what allows a deduction theorem to be maintained and prevents such atrocities as the deduction of Gp from p .

Let L_A be the set of all A -theorems. As all \mathcal{L} instances of PC1-PC3 and TL1-TL4 are axioms and $MP_{\mathcal{L}}$, $HGen_{\mathcal{L}}$ and $GGen_{\mathcal{L}}$ are rules of inference, we have that L_A is a tense logic. It is easy to see that $\phi \in L_A$ iff ϕ is a theorem of A ; that is, L_A -derivability and A -theoremhood coincide. We are thus entitled to a bit of notational abuse and write $\vdash_A \phi$ to indicate that ϕ is an A -theorem. In a similar vein it is easy to see that $\Sigma \vdash_{L_A} \phi$ iff there is an A -proof of ϕ from assumptions Σ , so again we usually write $\Sigma \vdash_A \phi$ to indicate the existence of this proof. Finally, let L be a tense logic and A an axiomatisation, both in language \mathcal{L} . Then A is an axiomatisation of L iff $L = L_A$. In this chapter and the next we will be interested in axiomatising certain interesting logics, such as the logic consisting of all NTL validities, $Th(\mathcal{U})$, or the logic of SPOs, $\bigcap \{Th(T) : T \text{ is a SPO}\}$. In what follows, whenever it is convenient we conflate the ‘logic perspective’ talk and the ‘axiomatic perspective’ talk. For example we often talk of an axiomatisation A being sound and complete with respect to some class of frames \mathcal{T} — by this we mean that L_A has these properties.

Modal logics must contain all instances of $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$, and modal axiomatisations have all instances as axioms. Similarly, if L is a modal logic and $\phi \in L$, then $\Box\phi \in L$; and modal axiomatisations have available $NEC_{\mathcal{L}} = \{\langle \phi, \Box\phi \rangle : \phi \in WFF_{\mathcal{L}}\}$, the rule of necessitation.

4.2 The Axiomatisation K_{nt}

Our first goal is to axiomatise the minimal logic, $Th(\mathcal{U})$. We already have some information about this logic: from the correspondence considerations of the previous chapter we know that $Th(\mathcal{U})$ is recursively enumerable, and from the filtration argument of Chapter 1 we know that it is recursive as well — so the fact that it is axiomatisable will come as no surprise. What is pleasant is the neat way this can be done: the minimal logic can be axiomatized by the addition of a single schema to K_t , the usual axiomatisation of the minimal Priorian tense logic. The requisite schema is called the NOM schema, and to present it we need a new piece of notation. For any language \mathcal{L} let an *existential tense* be any unbroken sequence of P s and F s. The sequence may contain both P s and F s, and we regard the null sequence as an existential tense. Thus $FPPFP$, F and PPP are existential tenses; $PFFGPP$ isn't because it contains a universal operator, G . We normally use E , E' , and so on as metavariables across existential tenses. By the length of an existential tense we mean the sequence length. We will later use *universal tenses*. By this is meant any unbroken, possibly mixed, sequence of G s and H s, including the null sequence. A , A' , A'' ... are used as metavariables over universal tenses, and again by the length of such a tense we mean the sequence length.

We now present the axiomatisation K_{nt} . Suppose we are working in a fixed language \mathcal{L} . Let ϕ, ψ and θ be metavariables over the wffs of \mathcal{L} , E and E' be metavariables over the existential tenses of \mathcal{L} , and n be a metavariable over the nominals of \mathcal{L} . (Of course if \mathcal{L} is a purely Priorian language, n ranges over the empty set.) Then by K_{nt} in the language \mathcal{L} is meant K_t augmented by all \mathcal{L} instances of the following schema:

$$\text{NOM} \quad E(n \wedge \phi) \wedge E'(n \wedge \psi) \rightarrow E(n \wedge \phi \wedge \psi)$$

The rules are those of modus ponens and temporal generalisation. Note that if \mathcal{L} is a purely Priorian language NOM has no instances, and we are left with just instances in variables of PC1-PC3, and TL1-TL4; that is, K_{nt} collapses into K_t .

NOM is thus the only new item needed to control the nominals and their interaction with other wffs. What does the NOM schema say? Let's instantiate it in i and consider:

$$E(i \wedge \phi) \wedge E'(i \wedge \psi) \rightarrow E(i \wedge \phi \wedge \psi).$$

Think of the points of a model as boxes holding items of information. Suppose we are standing at a point t in some frame \mathbf{T} and we know that both $E(i \wedge \phi)$ and $E'(i \wedge \psi)$ are true. This means we know that if we follow a certain zig-zag path from t , (the one coded up by E), we can get to a box marked i and containing the information ϕ ; and that if we follow another possibly different path from t , (the one coded up by E') we get to another box, also marked i , and containing the information ψ . *But there is only one box marked i .* Hence this single box contains the both the information ϕ and the information ψ , and the paths coded for by E and E' lead to the same point. This is precisely what the consequent of NOM gives us. In a nutshell, the NOM schema consists of all the path equations that must be satisfied in any model.

Theorem 4.2.1 (Soundness) $\vdash_{K_{nt}} \phi$ implies $\models \phi$

Proof:

There is nothing new to prove concerning the components from K_t . The only work lies with NOM, and it is straightforward to turn the 'box and path' thought experiment into a proof of the soundness of this schema. \square

4.3 Completeness of K_{nt}

We wish to prove that K_{nt} is strongly complete with respect to \mathcal{U} , hence it suffices to show that every consistent ¹ set of sentences has a model. How can this be done? The first idea

¹For the remainder of this section ' K_{nt} ' qualifications on terminology will be dropped wherever possible: consistency, derivability and MCS for example, will mean K_{nt} -consistency, K_{nt} -derivability and K_{nt} -MCS respectively; and we write \vdash instead of $\vdash_{K_{nt}}$.

that springs to mind is to use the *canonical Henkin model* method, familiar from standard modal and tense logic. In this approach one defines the *total Henkin frame* $\mathbf{H} = \langle H, <_h \rangle$, where H is the set of all MCS and $<_h$ is the *usual ordering on Henkin frames*, defined by $h <_h h'$ iff for all wffs ϕ , $G\phi \in h$ implies $\phi \in h'$. One then forms the model $\mathbf{C} = \langle \mathbf{H}, V \rangle$, where V is the *natural valuation* on H . That is $V(p) = \{h \in H : p \in h\}$, for all variables p . It is now routine to verify by induction that for any wff ϕ and any $h \in H$, $\mathbf{C} \models \phi[h]$ iff $\phi \in h$, a result sometimes called the Fundamental Theorem of normal modal logics. This immediately shows that any consistent set of sentences has a model. For let Σ be any such set. Form Σ^∞ . As Σ^∞ is an MCS it is an element of H , and by the previous remark $\mathbf{C} \models \sigma[\Sigma^\infty]$, for all $\sigma \in \Sigma^\infty$; or simply $\mathbf{C} \models \Sigma^\infty[\Sigma^\infty]$. Thus all the sentences in Σ are true in \mathbf{C} at Σ^∞ as $\Sigma \subseteq \Sigma^\infty$. This construction thus solves the model existence problem for the minimal logics — and, indeed, for most commonly encountered logics — in standard tense and modal languages. Note, moreover that it demonstrates model existence in very efficient fashion — \mathbf{C} has the property of *canonicity*:

Definition 4.3.1 *Let L be a logic. A model \mathbf{M} is L -canonical iff for every L -consistent set of wffs Σ there is a point t in \mathbf{M} such that $\mathbf{M} \models \Sigma[t]$. \square*

That \mathbf{C} is canonical is obvious: canonicity is engineered into its construction.

As soon as we try to extend this construction to languages with nominals we run into a problem. Forming \mathbf{H} is unproblematic — the total Henkin frame is a perfectly good frame, even when dealing with languages with nominals — but the natural mapping $V : \text{ATOM} \rightarrow \text{Pow}(H)$ defined by $V(a) = \{h \in H : a \in h\}$ is *not* a valuation. The problem, of course, lies with the clause that nominals must be assigned singleton subsets of H : V does not ^{do} this. This can be seen as a corollary of the proof of the following general result:

Theorem 4.3.1 (Non-existence of Canonical Models) *If L is a language containing at least one nominal i then there are no K_{nt} -canonical models.*

Proof:

Neither $\neg(i \wedge Fi)$ nor $\neg(i \wedge \neg Fi)$ is valid, hence by soundness neither is K_{nt} -derivable. Thus both $i \wedge Fi$ and $i \wedge \neg Fi$ are consistent. But in any model M , i is true at exactly one point, so it is impossible to have both wffs true in M . Thus no model is canonical. \square

In terms of the canonical Henkin model construction this shows that i will be contained in many distinct MCS: thus V is not a valuation and the outlined method of proving a model existence lemma vanishes. The crux of the matter is simply that H is far too big: it contains too many MCS. We need to be a little more delicate if we wish to use Henkin methods to build our models. We now present two Henkin completeness proofs. The first is a step by step Henkin construction process that yields countable models; the second is a more powerful method that generally gives uncountable models.

4.3.1 First Proof: Inductive construction of a Henkin frame

The heart of the first proof method is due to David Makinson [62], and was one of the first means by which Henkin techniques were applied in modal logic. The method was later used in Robert McArthur's textbook on tense logic [63]. Makinson's method is best approached by considering Henkin proofs for propositional languages. Such completeness proofs are extremely simple: given Σ , form Σ^∞ ; define the natural (propositional) valuation — and immediately we have that $V(\phi) = 1$ iff $\phi \in \Sigma^\infty$. Note the intuition: Σ^∞ is a 'time slice', and a sentence is true iff this time supports the fact ϕ . A Henkin proof for propositional calculus is essentially a natural way of building a single verifying time. Now suppose that our language is a standard tense logical language. Our initial Σ^∞ may well contain sentences of the form $F\phi$ or $P\phi$. Such sentences place a constraint on other times; they demand the existence of a future or past time in which ϕ holds. In Henkin terms this means that for each sentence $F\phi$ we must make an MCS that verifies ϕ and which is later or earlier than Σ^∞ in some appropriate ordering on MCS. Makinson's solution was to form *kernel futures*, the 'seeds' from which such future worlds can

grow.² If $F\phi \in \Sigma^\infty$ then the future kernel containing ϕ , K_ϕ , is the set of sentences consisting of ϕ together with all formula ψ such that $G\psi \in \Sigma^\infty$. We then extend K_ϕ to K_ϕ^∞ , and by construction we have a new world verifying ϕ . Note that we don't add *all* the MCS extending K_ϕ , just the single MCS K_ϕ^∞ . If we added all extending MCS, a countable model could not be guaranteed, even for countable languages. Further note that everything that was always going to be the case at Σ^∞ , actually *is* the case in K_ϕ^∞ , and so intuitively Σ^∞ precedes K_ϕ^∞ . This process is then iterated — we extend *all* future kernels of *all* the MCSs — then all the MCSs thus produced are collected together to form the points of the Makinson frame, the points are ordered as usual, and the natural valuation is defined, giving a model. What is of interest for our purposes is that even when languages with nominals are used, the natural mapping is a weak valuation: every nominal occurs in at most one MCS.

Intuitions given, let's begin. I'll first note without proof that the rule of Replacement of Equivalentents is a derived rule in K_{nt} . That is, if $\vdash \phi$ and $\vdash \psi \leftrightarrow \theta$, then $\vdash \phi[\psi/\theta_n]$, where this formula is the result of replacing the n^{th} occurrence of θ in ϕ by ψ . Further, for any existential tense E and any wffs ϕ and ψ we have that $\vdash \neg E(\phi \wedge \neg\phi)$, and $\vdash E(\phi \wedge \psi) \rightarrow (E\phi \wedge E\psi)$. Proof of the following lemma may be found in [63] or [46]:

Lemma 4.3.1 (Operator Additivity Lemma) *Let Σ be a set of sentences. Let $\Sigma^{G^+} = \{G\phi : \phi \in \Sigma\}$ and $\Sigma^{H^+} = \{H\phi : \phi \in \Sigma\}$. Then $\Sigma \vdash \phi$ implies $\Sigma^{G^+} \vdash G\phi$ and $\Sigma^{H^+} \vdash H\phi$.*
□

We now define the entities used in the construction and note some of their properties. Let Σ be an MCS. Define Σ^{G^-} to be $\{\phi : G\phi \in \Sigma\}$, and Σ^{H^-} to be $\{\phi : H\phi \in \Sigma\}$. Then for each $F\phi \in \Sigma$ we call the set $\Sigma^{G^-} \cup \{\phi\}$ a *kernel future* of Σ ; and for any $P\phi \in \Sigma$ we call the set $\Sigma^{H^-} \cup \{\phi\}$ a *kernel past* of Σ . If Σ' is a kernel future (past) of Σ we write $\Sigma'KerF\Sigma$, ($\Sigma'KerP\Sigma$).

²Makinson does not use this terminology. The present presentation, is loosely based around McArthur's, though this particular piece of terminology is my own.

Lemma 4.3.2 (Kernel Consistency Lemma) *If Σ is an MCS then any kernel past or future of Σ is consistent.*

Proof:

Consider the case for kernel futures. Suppose Σ is an MCS and $F\phi \in \Sigma$. We wish to show that $\Sigma^{G^-} \cup \{\phi\}$ is consistent, so suppose for the sake of contradiction that it isn't. Then $\Sigma^{G^-} \cup \{\phi\} \vdash \perp$, and thus $\Sigma^{G^-} \vdash \neg\phi$. But as proofs are of finite length we can find $\{\psi_1, \dots, \psi_n\} \subseteq \Sigma^{G^-}$ such that $\{\psi_1, \dots, \psi_n\} \vdash \neg\phi$; but then by the Operator Additivity Lemma we have that $\{G\psi_1, \dots, G\psi_n\} \vdash G\neg\phi$. Hence $\Sigma \vdash G\neg\phi$ as $\{G\psi_1, \dots, G\psi_n\} \subseteq \Sigma$. But $G\neg\phi$ is just $\neg F\phi$, and as $\Sigma \vdash F\phi$ we have a contradiction. \square

Kernel pasts and futures are related as we would hope: if $\Sigma \text{Ker} F\Sigma'$ then $H\phi \in \Sigma^\infty$ implies $\phi \in \Sigma'$; and if $\Sigma \text{Ker} P\Sigma'$ then $G\phi \in \Sigma^\infty$ implies $\phi \in \Sigma'$. The proofs are analogous to that of the Order Equivalence Lemma, given below.

We are now ready for the construction. Given a consistent set of sentences Σ , proceed as follows. Define:

$$\begin{aligned} T_0 &= \{\Sigma^\infty\} \\ T_{n+1} &= T_n \\ &\quad \cup \{f^\infty : f \text{Ker} Ft \text{ and } t \in T_n\} \\ &\quad \cup \{p^\infty : p \text{Ker} Pt \text{ and } t \in T_n\} \\ T &= \bigcup_{n < \omega} T_n \end{aligned}$$

We denote $\{\Sigma^\infty\}$ by t_0 . Now define a relation $<_t$ on T in the manner usual for tense logic Henkin constructions: $t <_t t'$ iff for all wffs ϕ , $G\phi \in t$ implies $\phi \in t'$. Thus we have a Henkin frame $\mathbf{T} = \langle T, <_t \rangle$.

Lemma 4.3.3 (Order Equivalence Lemma) *For all $t, t' \in T$, $t <_t t'$ is equivalent to any of the three following conditions: $H\phi \in t'$ implies $\phi \in t$; $\phi \in t'$ implies $F\phi \in t$; or $\phi \in t$ implies $P\phi \in t'$, for all wffs ϕ .*

Proof:

We prove the first case.

(\Leftarrow). Suppose $H\psi \in t'$ implies $\psi \in t$. We need to show $G\phi \in t$ implies $\phi \in t'$. So suppose $\phi \notin t'$. Therefore $\neg\phi \in t'$ as t' is an MCS. Therefore $HF\neg\phi \in t'$, by TL4. Therefore $F\neg\phi \in t$ by our initial assumption. But then $\neg F\neg\phi \notin t$ as t is also an MCS. That is, $G\phi \notin t$ and so we have shown the contrapositive of the desired result.

(\Rightarrow). So now suppose that $t <_t t'$, that is, $G\psi \in t$ implies $\psi \in t'$. Suppose $\phi \notin t$. Therefore $\neg\phi \in t$. Therefore $GP\neg\phi \in t$ by TL3. So by our initial assumption, $P\neg\phi \in t'$. Therefore $\neg P\neg\phi \notin t'$ —that is $H\phi \notin t'$ —and again we have shown the contrapositive. \square

It follows from this that $pKerPt'$ implies $p^\infty <_t t'$. That $fKerFt'$ implies $t' <_t f^\infty$ is immediate from the definition of $<_t$.

If we were dealing with a purely Priorean language we would be through at this stage: we could simply define the natural valuation V and the usual induction would show that for all $t \in T$, $\langle\langle T, <_t \rangle, V\rangle \models \phi[t]$ iff $\phi \in t$ and thus that $\Sigma \subseteq t_0$ was satisfied at t_0 . For languages with nominals, however, we must do a little more work: we need to demonstrate the existence of a natural mapping that assigns singletons to nominals. This process falls into two stages. Firstly we demonstrate that the natural mapping V on the frame $\langle T, <_t \rangle$ *almost* works, for using the NOM schema we can demonstrate that every nominal is contained in *at most one* $t \in T$. However, as we shall see, we can't rule out the possibility that some nominal i is contained in no element of t . To cater for such nominals we are going to have to extend our frame; we turn to this matter later.

In order to prove the existence of a weak valuation we need to take a closer look at the structure of the Makinson frame. The following machinery will prove useful: we define a rank function $R : T \rightarrow N$ by:

$$\begin{aligned} R(t_0) &= 0 \\ R(t) &= n > 0 \text{ if } t \neq t_0 \text{ and } t \in T_n \text{ and } t \notin T_{n-1}. \end{aligned}$$

and a relation $IC \subseteq T^2$ by $t'ICt$ iff $R(t') = R(t) + 1$ and there is an $v \subseteq t'$ such that $vKerFt$ or $vKerPt$ and $t' = v^\infty$. If $t'ICt$ we say t' is *immediately conjoined* to t . Note that IC is irreflexive; that $R(t) = 0$ iff $t = t_0$; and that $R(t) > 0$ implies there is a $t' \in T$

such that $tICt'$ and $R(t') = n - 1$. We can now prove the following lemma by induction on $R(t)$:

Lemma 4.3.4 (Path Lemma) *If $t \in T$, and $R(t) = n > 0$ then there exists a sequence $\langle t^x \rangle_{0 \leq x \leq n}$ such that $t^0 = t_0$, $t^n = t$, and $t^{x+1}ICt^x$. (Furthermore, $R(t^x) = x$.)* \square

This result need not hold for $n = 0$ as t_0 may have no kernel past or futures. Note that the Path Lemma shows that T is connected: given two distinct points in the frame we can always find a zig-zag path from one to the other, for by the last lemma we can always zig-zag back to t_0 from any other point, so, given two distinct points in the frame, we can zig-zag from one to the other via t_0 . There may be other shorter paths, but at least there is always this route.

Lemma 4.3.5 (Existential Tense Lemma) *For all $t \in T$, and all $\phi \in t$, there exists an existential tense E such that $E\phi \in t_0$. (Furthermore, $\text{len}(E) = R(t)$.)*

Proof:

Induction on $R(t)$. The base case is trivial: if $R(t) = 0$ then $t = t_0$, and all elements $\phi \in t_0$ are preceded by the null tense. So suppose the result holds for all t' such that $0 < R(t') < n$. That is, $\psi \in t'$ implies $E\psi \in t_0$, for some existential tense E , (and $\text{len}(E) = R(t')$). Suppose $R(t) = n$. By the second property we noted for R , there exists t'' such that $R(t'') = n - 1$ and $tICt''$. Now suppose $\phi \in t$. Then by TL3 and TL4, $HF\phi$ and $GP\phi \in t$. But as $tICt''$, by previous lemmas we have $t < t''$ or $t'' < t$, and so either by the definition of $<$ or the Order Equivalence Lemma we have that $P\phi$ or $F\phi \in t''$. But then by the inductive hypothesis, $EP\phi$ or $EF\phi \in t_0$, (and $\text{len}(E) = R(t'') = n - 1$). But both EP and EF are existential tenses of length n . \square

We can now use the NOM schema together with the Existential Tense Lemma to prove the crucial lemma. First some notation: if Σ is a set of sentences, we define $\text{Nom}(\Sigma)$ to be $\{i \in \text{NOM}_\ell : i \in \Sigma\}$. Then we have:

Lemma 4.3.6 (Unique Occurrence Lemma) *For all $t, t' \in T$, $t \neq t'$ implies $\text{Nom}(t) \cap \text{Nom}(t') = \emptyset$*

Proof:

As t and t' are distinct MCS there exists $\phi \in t$ such that $\neg\phi \in t'$. Assume they share a nominal in common, say i . Then by the existential tense lemma there are existential tenses E and E' such that $E(i \wedge \phi) \in t_0$ and $E'(i \wedge \neg\phi) \in t_0$. Hence by NOM, $E(i \wedge \phi \wedge \neg\phi) \in t_0$, and thus $E(\phi \wedge \neg\phi) \in t_0$. But we also know that $\vdash \neg E(\phi \wedge \neg\phi)$ for any existential tense E , and hence $\neg E(\phi \wedge \neg\phi) \in t_0$ —contradicting t_0 's consistency. \square

So the ‘at most one’ clause for our proposed natural valuation is established; the natural mapping on the Makinson frame is a weak valuation.

There still remains a minor problem with the ‘at least one’ part: it can very easily happen that some nominal, say i , is not in any MCS in \mathbf{T} . For example, consider the (consistent) set of sentences $\{\neg E i : E \in ET\}$, where i is some nominal. If this was our original set of sentences Σ , then it is clear that Makinson’s construction will force $\neg i$ into every $t \in T$, and thus for all $t \in T$, $i \notin t$. Worse still, consider the set of sentences $\{\neg E n : n \in NOM_{\mathcal{L}} \text{ and } E \in ET\}$. This has the effect of ‘forcing out’ *all* the nominals: only negated nominals occur at any point in T ! Because of such possibilities we cannot guarantee that the weak valuation is also a valuation.

But the remedy is clear: we enlarge $\langle T, <_t \rangle$ by adjoining to it another frame $\langle U, <_u \rangle$ to form a new composite frame $\langle S, <_s \rangle$. We do this in such a way that the two subframes, \mathbf{T} and \mathbf{U} of \mathbf{S} , are disconnected. We can then safely assign all ‘unused’ nominals singleton subsets of U . This will complete our natural valuation, and because the two subframes are disconnected, the assignments we make in \mathbf{U} won’t affect what is going on in \mathbf{T} : Σ will still be satisfied at t_0 .

There are several ways we could extend \mathbf{T} . Simplest of all is to adjoin to \mathbf{T} a frame \mathbf{U} consisting of a singleton irreflexive point t_{-1} unrelated to any point in T — a ‘point at infinity’ — and assign all unused nominals $\{t_{-1}\}$. Another option is to form the disjoint union of \mathbf{T} with itself, identify \mathbf{T} with (say) the left projection of $\mathbf{T} \uplus \mathbf{T}$, and assign all unused nominals the point t_0 in the right projection of compound frame. Either of these methods immediately yields a suitable model, our desired strong completeness theorem and thus concludes the proof. However let’s pause and examine a more elegant third

method: to *carefully* choose some new consistent set of sentences Σ_u and generate a new frame U by the Makinson method.

The intuition is simple. We are worried about nominals not assigned points in T : so simply choose a set of sentences Σ_u consisting of all such unassigned nominals and sentences which will explicitly ‘force out’ all the nominals that *do* occur in some $t \in T$ from the new inductively generated frame. Note that we know how to ‘force out’ nominals: the sets of sentences we noted above as giving rise to the possibility of unassigned nominals perform the task wonderfully: such sets of sentences caused the problem in the first place and now will be used to fix it! So let $Nom(T) = \bigcup_{t \in T} Nom(t)$. Define:

$$\Sigma_u = NOM_L \setminus Nom(T) \cup \{\neg Ei : i \in Nom(T) \text{ and } E \in ET\}.$$

Σ_u is consistent as it has a model.³ So we can form Σ_u^∞ and form a Makinson frame $U = \langle U, <_u \rangle$, with Σ_u^∞ as the initial point u_0 . As U is a Makinson frame, results analogous to those proved for T hold for U also: most importantly this includes the Unique Occurrence Lemma.

We leave the simple proof that $T \cap U = \emptyset$ to the reader. Let $S = T \cup U$ and $<_s = <_t \cup <_u$. Then the frame $S = \langle S, <_s \rangle$ is a countable Henkin frame consisting of two generated subframes as required. Furthermore, as any nominal occurring in T occurs only once there and does not occur in U , and as any nominal not occurring in T occurs precisely once in U (at u_0), the natural mapping V on S is a valuation and we are through.⁴ Thus we have by the Strong Completeness Lemma that:

³For example, if P is the frame $\langle \{1, 2\}, \emptyset \rangle$ and V_P is any valuation on P satisfying $V(i) = \{1\}$, for all $i \in NOM_L \setminus Nom(T)$ and $V(i) = \{2\}$ otherwise, then $\langle P, V_P \rangle \models \Sigma_u[1]$.

⁴The standard induction that shows that $\langle S, V \rangle$ suffices runs as follows. The atomic case is automatic as S is a Henkin frame, and the boolean cases are straightforward. So suppose that $\phi = F\psi$. Then $V(F\psi, s) = 1$ implies $\exists s' > s \ V(\psi, s') = 1$, which implies $\psi \in s'$ by the inductive Hypothesis, which in turn implies that $HF\psi \in s'$. But then by the Order Equivalence Lemma, as $s < s'$, $F\psi \in s$.

Theorem 4.3.2 (Strong Completeness) $\Sigma \models_m^{\mathcal{U}} \phi$ implies $\Sigma \vdash \phi$. \square

This in turn gives a weak completeness theorem and shows that K_{nt} is characterised by the class of all frames; or, to put it another way, that our axiomatisation yields all and only the NTL validities.

The above proof also yields a strong completeness theorem for languages of weak NTL — we merely make immediate use of the weak valuation guaranteed to exist at the end of the first construction stage; that is, we don't augment our Henkin frame — hence K_{nt} axiomatises the set of validities for such languages. As a corollary we have that NTL \mathcal{L} and its corresponding weak language, \mathcal{L}^W , have the same set of validities. Although \mathcal{L} is more expressive than \mathcal{L}^W , as we know from our discussion of disjoint unions, this does not show on \mathcal{U} .

Note further that K_{nt} also yields all the validities of strong languages of NTL with at least countably many nominals; this is immediate as we know from Chapter 1 that the set of ordinary and strong NTL validities coincide for such languages. Note, however, that we only get a weak completeness theorem by this argument; nothing in the above construction guarantees that every point of S contains a nominal.

4.3.2 Second proof: Generated subframes of H

As any Makinson frame is a subframe of the canonical Henkin frame H , it is natural to enquire what it is about Makinson frames that guarantees that the unique occurrence property holds for them while it fails for H as a whole. An examination of the sequence of lemmas leading to the proof of the Unique Occurrence Lemma makes it apparent that the crucial fact is that Makinson frames are connected: the NOM schema can appropriately

Conversely, suppose $F\psi \in s$. Let $\Sigma_\psi = \{\psi\} \cup s^{G-}$. Then $\Sigma_\psi \text{Ker } Fs$, and by the construction of S , $\Sigma_\psi^\infty \in S$. As $\psi \in \Sigma_\psi^\infty$, by the induction hypothesis $V(\psi, \Sigma_\psi^\infty) = 1$. But $s <_s \Sigma_\psi^\infty$ by construction, hence $V(F\psi, s) = 1$. The case for $P\psi$ is analogous.

regulate the occurrence of nominals in subsets of \mathbf{H} , but its ability to do so depends on the existence of a path between any two points. Once this has been seen, a very simple completeness proof becomes possible: we can use subframes of \mathbf{H} generated from points to make our model.

We now present such a proof. Instead of directly using NOM we will use the fact that we have all instances of the following schema at our disposal:

$$\text{SWEEP}_w \quad n \wedge \phi \rightarrow A(n \rightarrow \phi).$$

Here A is a metavariable over universal tenses, n a metavariable over nominals, and ϕ a metavariable over wffs; thus typical instances of SWEEP_w are $i \wedge \phi \rightarrow HHGH(i \rightarrow \phi)$, and $i \wedge \phi \rightarrow (i \rightarrow \phi)$. As all instances of SWEEP_w are valid, by the previous completeness theorem all instances of SWEEP_w must be theorems of K_{nt} — and later in the chapter we show how to construct syntactic proofs — but the proof of the lemma given below shows that the SWEEP_w schema could replace NOM in our axiomatisation of $Th(\mathcal{U})$: that is, we have found a second minimal axiomatic basis. We proceed straight to the crucial lemma:

Lemma 4.3.7 (Unique Occurrence Lemma) *Let Σ be an MCS, and S be the subframe of \mathbf{H} generated by Σ . Then for all $s, s' \in S$, if some nominal i is in both s and s' then $s = s'$.*

Proof:

Suppose two points s and s' in S contain the same nominal i . As S is generated from a single point Σ it is connected, and thus there is a path between s and s' . Let n be the length of this path, and let $A^{(s \rightarrow s')}$ be the universal tense that corresponds to the path as seen from s . (That is, starting at s we traverse the path until we reach s' , writing down a G for every move forward in time, and H for every move backwards.) As all instances of SWEEP_w occur in s , then in particular we have that

$$i \wedge \phi \rightarrow A^{(s \rightarrow s')}(i \rightarrow \phi) \in s.$$

But as $i \in s$, then for all $\phi \in s$ we have by modus ponens that $A^{(s \rightarrow s')}(i \rightarrow \phi) \in s$. But then by n applications of the definition of $<_A$ and the Order Equivalence Lemma we have that $i \rightarrow \phi \in s'$, and as $i \in s'$ we have by modus ponens that $\phi \in s'$. As s and s' are MCS this means that $s = s'$. \square

As the Unique Occurrence Lemma holds for every point-generated subframe of \mathbf{H} , we can ‘double generate’ a verifying model for any consistent set of sentences Σ . That is, given such a set Σ , form Σ^∞ , and define Σ_u as we did in the Makinson proof. But then taking as our frame the subframe of \mathbf{H} generated from the doubleton $\{\Sigma^\infty, \Sigma_u^\infty\}$ yields our verifying model: the definition of Σ_u guarantees that this frame will consist of two disconnected islands, the Unique Occurrence Lemma holds for each island, and by design every nominal occurs in precisely one MCS in one of the islands and we have our completeness result. This double generation technique will be assumed in later chapters as our basic model building technique.

Note that there is a subtle difference between the two proofs we have given of the Unique Occurrence Lemmas. In the Makinson proof we used a ‘three point argument’, whereas the proof above uses a ‘two point argument’. The argument as presented in the Makinson proof was presented from the viewpoint of an observer standing at t_0 . The argument took the form, ‘Suppose, on looking out from t_0 , we see that there are two distinct points t and t' out in the frame sharing some nominal $i \dots$ ’; and went on from there to show what impact this would have on the privileged viewing point t_0 . In the above argument, on the other hand, no use is made of the privileged point from which the frame was generated: the argument merely appeals to the fact that a path must exist between t and t' . Now clearly we could have used a two point argument in the Makinson proof; and a little thought will reveal that if we had used such a proof we wouldn’t have needed all instances of NOM, but merely all instances of the following weak NOM schema:

$$\text{NOM}_w \quad n \wedge E(n \wedge \phi) \rightarrow \phi.$$

Similarly, we could have used a ‘three point argument’ on the subframe of \mathbf{H} generated

by Σ^∞ , with Σ^∞ as the privileged point, and then the following schema is a natural basis:

$$\text{SWEEP} \quad E(n \wedge \phi) \rightarrow A(n \rightarrow \phi).$$

The main interest lies in the fact that in proving completeness for modal languages we *cannot* use two point arguments — we have to make use of three point arguments. As we shall see later, modal languages require analogs of NOM or SWEEP; using merely NOM_w or SWEEP_w does not axiomatise the minimal logic; this is yet another manifestation of the fact that with modal languages we can only look forwards, not back. In the meantime it should be clear that as far as tensed languages with nominals are concerned, we now have four schema which can be added to K_t to axiomatise $Th(\mathcal{U})$: NOM, NOM_w , SWEEP and SWEEP_w .

4.4 Some K_{nt} -theorems

In this section we examine some theorems of the minimal logic and give condensed syntactical proofs of them. Proofs are set out in four columns in a fashion that mimics linear natural deduction systems such as those of Lemmon [59] or Copi [21]. The first column simply contains a number, identifying the step of the proof; the second contains the actual wff corresponding to that step; the third contains the justification of the step; and the fourth is a (possibly empty) list of numbers, recording the assumptions on which that step depends. The justification column contains an annotation and (possibly) some numbers. If no numbers are present the line is either an assumption, a theorem, or an axiom. If the annotation on such a line says ‘assumption’, that’s exactly what the wff in column 2 is; if it says something else (such as NOM), the line is an instance of the named theorem or axiom. The numbers (if present) state which earlier line(s) that line was deduced from, and the annotation tells us what the inference step was. At its most unhelpful such an annotation will merely say PC or TL; that is, it just indicates whether the inference was propositional or tense logical. This is only used when the inference is very simple; usually the annotations are more specific. MP means modus ponens. Inst (for instantiation) annotates such tense logical moves as the deduction of $F(\phi \wedge \psi)$ from

earlier occurrences of $G\phi$ and $F\psi$. TMP stands for temporal modus ponens, and records an application of modus ponens under the scope of a block of universal or existential tenses. For example, the deduction of $E\psi$ from $E(\phi \rightarrow \psi)$ and $\vdash \phi$ is annotated as TMP.

A good place to start looking for validities is in the way nominals interact with universal tenses. If Gi is true at some point t there are only three possibilities: t has no successors at all, t is its own unique successor, or t has a unique successor t' and $t' \neq t$. In short, nominals ‘strongly interact’ with universal tenses: such expressions can be true only in a very limited set of circumstances — and, conversely, when such circumstances *do* occur, we know a lot about the way the truth of other wffs is locally distributed. I call theorems expressing such information *end effects*. An example is $i \wedge Gi \wedge F\phi \rightarrow \phi$, which can be proved as follows:

1.	$i \wedge Gi \wedge F\phi$	Assumption	
2.	$F(i \wedge \phi)$	1; Inst	1
3.	$i \wedge F(i \wedge \phi)$	1, 2; PC	1
4.	$i \wedge F(i \wedge \phi) \rightarrow \phi$	NOM _W	1
5.	ϕ	3, 4; MP	1
6.	$i \wedge Gi \wedge F\phi \rightarrow \phi$	1, 5; Discharge	

Note that NOM_W was used. Indeed one of our Unique Occurrence schemas had to be used: if a proof was possible without their use then by substituting new variables for uniformly throughout the derivation we would have a K_t proof of $p \wedge Gp \wedge F\psi \rightarrow \psi$, where ψ (and hence the whole formula) is purely Priorean. But this schema is not valid and hence not a theorem of K_t .

Next consider the following theorems and nontheorems of minimal Priorean tense logic. All six forward looking schemas expressing merges between universal and existentially quantified information are listed:

1. $\vdash_{K_i} G\phi \wedge G\psi \rightarrow G(\phi \wedge \psi)$
2. $\vdash_{K_i} G\phi \wedge F\psi \rightarrow F(\phi \wedge \psi)$
3. $\nvdash_{K_i} G\phi \wedge F\psi \rightarrow G(\phi \wedge \psi)$
4. $\nvdash_{K_i} F\phi \wedge F\psi \rightarrow F(\phi \wedge \psi)$
5. $\nvdash_{K_i} G\phi \wedge G\psi \rightarrow F(\phi \wedge \psi)$
6. $\nvdash_{K_i} F\phi \wedge F\psi \rightarrow G(\phi \wedge \psi)$

What happens when nominals are substituted for ϕ or ψ or both? Clearly the two valid schemas remain valid, and equally clearly schemas 5 and 6 are irredeemable. Schema 4 has valid subschemas, but we already know about these: they're instances of NOM. But 3, however, does yield something new: $Gn \wedge F\psi \rightarrow G(n \wedge \psi)$. This subschema is another end effect: under 'end conditions' we can turn existentially quantified information (here, ψ), into universally quantified information. Again NOM must be used to prove it:

1.	$\neg(Gi \wedge F\psi \rightarrow G(i \wedge \psi))$	Assumption	
2.	$Gi \wedge F\psi \wedge \neg G(i \wedge \psi)$	1; PC	1
3.	$Gi \wedge F\psi$	2; PC	1
4.	$F(i \wedge \psi)$	3; Inst	1
5.	$F\neg(i \wedge \psi)$	2; PC, $G = \neg F\neg$	1
6.	$Gi \wedge F\neg(i \wedge \psi)$	2, 5; PC	1
7.	$F(i \wedge \neg(i \wedge \psi))$	6; Inst	1
8.	$F(i \wedge \psi) \wedge F(i \wedge \neg(i \wedge \psi))$	4, 7; PC	1
9.	$F(i \wedge \psi \wedge \neg(i \wedge \psi))$	8; NOM, MP	1
10.	$F\perp$	9; PC	1
11.	$\neg F\perp$	Theorem	1
12.	\perp	10, 11; PC	1
13.	$\neg(Gi \wedge F\psi \rightarrow G(i \wedge \psi)) \rightarrow \perp$	1, 12; Discharge	
14.	$Gi \wedge F\psi \rightarrow G(i \wedge \psi)$	13; PC	

There are many other more or less obvious end effects, and all have simple generalisations to arbitrary existential and universal tenses; such investigations we leave to

the reader, and turn to another intuition: path exploration. This idea gives rise to two rather pretty schemas.

The idea behind the first schema is very simple. Suppose we are standing at some point t , and by following some path P we find ourselves back at t again. Then a reverse journey exists: we could traverse P back to front and still get to t . To make the notion of a reverse journey precise we need the notion of the *transposition* of an existential tense.

Definition 4.4.1 *The transposition of an existential tense E , E^T , is defined by letting the null tense be its own transpose, $F^T = P$, $P^T = F$, $(FE)^T = E^T P$, and $(PE)^T = E^T F$. That is, to transpose an existential tense we reverse the sequence and take its mirror image.* \square

Intuitively, if an existential tense E codes a path between points t and t' as seen by an observer at t , then E' codes the same path as viewed by an observer at t' . Note that $(E_1 E_2)^T = E_1^T E_2^T$. Tense transposition allows us $\xrightarrow{t \circ}$ transform between two observational viewpoints t and t' , as the following schema shows:

$$(\phi \wedge E\psi) \rightarrow E(\psi \wedge E^T\phi).$$

The schema states at t what t looks like to an observer at t' . That, at least, is the semantic intuition; as we'll be using the schema in the next two proofs it's important to note what it gives us proof theoretically. In these terms it's useful because it removes the block of tenses stuck on to the ψ and gives them wide scope, thus baring ψ . When ψ is some nominal i , the consequent of the formula is thus in the form required of antecedents of NOM. This move is crucial in the next two proofs. I call this schema Tense Transposition, or *TT* for short. Note that it is not peculiar to languages with nominals; all instances are standard validities and must therefore be theorems of ordinary tense logic. In fact it is easy to give an inductive specification of how to construct a K_{nt} proof of any desired

instance; this we relegate to a footnote.⁵ The schema asserting the existence of reverse journeys can now be given:

$$n \wedge En \rightarrow E^T n.$$

This is dubbed the Reverse Trip Schema. Note that its validity *does* hinge on the unique referring ability of nominals: instantiating n in variable p yields $p \wedge Ep \rightarrow E^T p$, which is invalid. We need to know where to return to, thus the starting point of the journey must be labelled, and this requires a nominal.

⁵Clearly we can display proofs for all instances where $\text{len}(E) = 0$, for such instances have the form $\phi \wedge \psi \rightarrow \psi \wedge \phi$. So suppose we can construct proofs of any instance where $\text{len}(E) = m$. Now all existential tenses of length $m + 1$ have the form PE or FE . Suppose E' is an existential tense of length $m + 1$ of the form PE . To construct a proof of $(\phi \wedge PE\psi) \rightarrow E(\psi \wedge (PE)^T \phi)$ proceed as follows:

1.	$\phi \wedge PE\psi$	Assumption	
2.	$\phi \rightarrow HF\phi$	TL4	1
3.	$HF\phi$	1, 2; MP	1
4.	$HF\phi \wedge PE\psi$	1, 3; PC	1
5.	$P(F\phi \wedge E\psi)$	4; Inst	1
6.	$(F\phi \wedge E\psi) \rightarrow E(\psi \wedge E^T F\phi)$	Theorem by Ind. Hyp.	1
7.	$P(E(\psi \wedge E^T F\phi))$	5, 6; TMP	1
8.	$PE(\psi \wedge (PE)^T \phi)$	7; Definition of T	1
9.	$(\phi \wedge PE\psi) \rightarrow PE(\psi \wedge (PE)^T \phi)$	1, 7; Discharge	

By inserting the proof of $(F\phi \wedge E\psi) \rightarrow E(\psi \wedge E^T F\phi)$, which by hypothesis we know how to construct, in place of line 6 in the above proof-scheme, we obtain a proof of the desired instance. The inductive proof-scheme for deriving all instances of $(\phi \wedge PE\psi) \rightarrow E(\psi \wedge (PE)^T \phi)$ is analogous to that above, and thus we have an inductive specification of the proof construction.

1. $i \wedge Ei$	Assumption	
2. $i \wedge Ei \rightarrow E(i \wedge E^T i)$	Tense Transposition	1
3. $E(i \wedge E^T i)$	1, 2; MP	1
4. $i \wedge E(i \wedge E^T i)$	1, 3; PC	1
5. $E^T i$	4; NOM_w , MP	1
6. $i \wedge Ei \rightarrow E^T i$	1, 5; Discharge	

This schema can be extended in a way that will prove useful in the next chapter. Suppose we are in a point labelled i and by following path P_1 we can reach a box containing information ψ . Moreover, suppose that on leaving this box we can follow another path, P_2 back to point i . Then as usual the reverse journey exists: but this time we can say something about the scenery encountered along the way. We can travel from i backwards along P_2 ; record the fact that ψ holds; and then travel backwards along P_1 to i : we have a second way of accessing ψ and returning with it. The schema expressing this is

$$i \wedge E_1(\psi \wedge E_2 i) \rightarrow E_2^T(\psi \wedge E_1^T i),$$

and is called the Stopover Schema.

1. $i \wedge E_1(\psi \wedge E_2 i)$	Assumption	
2. $E_1((\psi \wedge E_2 i) \wedge E_1^T i)$	1; Tense Transposition, MP	1
3. $E_1(E_2 i \wedge (\psi \wedge E_1^T i))$	2; PC	1
4. $E_1 E_2(i \wedge E_2^T(\psi \wedge E_1^T i))$	3; Tense Transposition, MP	1
5. $i \wedge E_1 E_2(i \wedge E_2^T(\psi \wedge E_1^T i))$	1, 4; PC	1
6. $E_2^T(\psi \wedge E_1^T i)$	1, 5; NOM_w , MP	1
7. $i \wedge E_1(\psi \wedge E_2 i) \rightarrow E_2^T(\psi \wedge E_1^T i)$	1, 6; Discharge	

Note the way the two applications of TT at lines 2 and 4 bare the i for subsequent feeding to NOM. The Stopover Schema will prove useful to us when we consider the logic of partial orders in the next chapter.

We conclude this section by considering the interderivability of the four different axiomatic bases we have for the minimal logic. In the following proofs we will often need

to form the mirror image of a universal or existential tense: the following notation will help keep things readable. Let E be an existential tense; $mi(E)$ will be denoted by \overline{E} — which of course is a *universal* tense. Similarly \overline{A} denotes the existential tense obtained by taking the mirror image of A . With the observation that $A = \neg\overline{A}\neg$, and $E = \neg\overline{E}\neg$, we are ready to derive all instances of SWEEP_W in K_{nt} :

1.	$\neg((i \wedge \phi) \rightarrow A(i \rightarrow \phi))$	Assumption	
2.	$(i \wedge \phi) \wedge \neg A(i \rightarrow \phi)$	1; PC	1
3.	$\overline{A}(i \wedge \neg\phi)$	2; $\neg A = \overline{A}\neg$, PC	1
4.	$(i \wedge \phi) \wedge \overline{A}(i \wedge \neg\phi)$	2, 3; PC	1
5.	$(i \wedge \phi \wedge \neg\phi)$	4; NOM, MP	1
6.	\perp	5; PC	1
7.	$(i \wedge \phi) \rightarrow A(i \rightarrow \phi)$	1, 6; Discharge, PC	

We next show how to derive all instances of SWEEP from SWEEP_W :

1.	$\neg(E(i \wedge \phi) \rightarrow A(i \rightarrow \phi))$	Assumption	
2.	$E(i \wedge \phi) \wedge \neg A(i \rightarrow \phi)$	1; PC	1
3.	$E(i \wedge \phi \wedge E^T \neg A(i \rightarrow \phi))$	2; Tense Transposition, MP	1
4.	$E(i \wedge \phi \wedge \neg \overline{E^T} A(i \rightarrow \phi))$	3; $E^T \neg = \neg \overline{E^T}$	1
5.	$(i \wedge \phi) \rightarrow \overline{E^T} A(i \rightarrow \phi)$	SWEEP_W	1
6.	$\overline{E}((i \wedge \phi) \rightarrow \overline{E^T} A(i \rightarrow \phi))$	5; HGen and GGen	1
7.	$E(\overline{E^T} A(i \rightarrow \phi) \wedge \neg \overline{E^T} A(i \rightarrow \phi))$	4, 6; TMP	1
8.	$E\perp$	7; PC	1
9.	\perp	8; TL	1
10.	$E(i \wedge \phi) \rightarrow A(i \rightarrow \phi)$	1, 9; Discharge, PC	

Note that TT was used at line three. As we shall see in our discussion of the minimal modal logic, this is not accidental; TT, or some equivalent *tense* logical schema must be appealed to if the strong schemas are to be derived from their weaker cousins.

The derivations of NOM from SWEEP , and SWEEP_W from NOM_W are both similarly straightforward.

4.5 Modal and multimodal languages

In this section we axiomatise the minimal logics for languages of NML, and multimodal languages with nominals; and examine the effect the addition of nominals has on the Henkin frame of the minimal modal logic.

Given what we already know, axiomatising $Th(\mathcal{U})$ for modal languages with nominals is easy. The modal analog of an existential tense is an existential modality, an unbroken sequence of \Diamond s; while the analog of a universal tense is a universal modality, an unbroken sequence of \Box s. With E and A now taken as ranging over such (possibly empty) sequences, the minimal logic in languages of NML (or weak or strong NML) can be axiomatised by adding as axioms all instances of the modalised version of either NOM or SWEEP to the minimal modal axiomatisation K .⁶ We assume some choice has been made and call the result K_{nm} . Completeness follows by either the Makinson argument — using ‘kernel possibilities’ — or the appeal to (modal) generated subframes of \mathbf{H} . This should be clear; for suppose we are standing at the ‘privileged point’ in the frame produced by either of these methods, that is, the point that is our original MCS Σ^∞ . In both constructions the privileged point t_0 has the following property: given any other t in the frame there is a modal path from t_0 to t . Thus any point in either of the frames can be ‘accessed’ from t_0 by a sequence of \Diamond s. But then the three point unique occurrence argument goes through: suppose that two distinct points t and t' have a nominal i in common. Then both $\Diamond^n(i \wedge \phi)$ and $\Diamond^m(i \wedge \neg\phi)$ are in t_0 for some n and m length sequences of \Diamond . Appealing to either NOM or SWEEP gives a contradiction.

However, if we only had NOM_W or $SWEEP_W$ to appeal to we *couldn't* prove the unique occurrence property. To see what could go wrong, imagine that both points t and t' as above are distinct from t_0 , and that there is no modal path from t to t' and vice-versa.

⁶ K has as axioms the usual propositional base $PC1 - PC3$, all instances of $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$, and the rules: $\vdash \phi \Rightarrow \vdash \Box\phi$, and modus ponens.

Suppose $i \wedge \phi$ occurs at t : then there simply is nothing in our axioms to block $i \wedge \neg\phi$ from occurring at t' . Because of the missing instances we can't look forward from t_0 and make the three point argument; and because of the inherent unidirectionality of modal languages we cannot 'look back' from t to t_0 and then 'round the corner' to t' — there are no tense transpositions in modal languages. In a nutshell: we lack the axioms for the three point argument, and two point arguments can't be made in modal languages. That, at any rate is the intuition; and it is easy to turn this intuition into a proof that neither of the axiomatic bases $K + \text{NOM}_W$ nor $K + \text{SWEEP}_W$ suffice to axiomatise the minimal nominal modal logic. We do this by showing that neither base can derive either the full NOM or SWEEP schemas. As is usual with such proofs, we proceed by finding a semantical property which distinguishes the derivable from the non-derivable wffs. This motivates the following:

Definition 4.5.1 *Let T be a frame and t and t' be distinct elements of T . We say t and t' are a separated pair iff there is no modal path from t to t' , and no modal path from t' to t . A frame is said to be separated iff it contains at least one separated pair.* \square

Note that this definition talks of *modal* paths, not zig-zag paths. We now change the interpretation of modal languages with nominals. Let \mathcal{L} be any language of nominal modal logic. In the *separated interpretation* for \mathcal{L} we define *separated valuations* on separated frames; in each separated valuation every nominal denotes exactly two distinct points, t and t' , where t and t' are a separated pair. Everything else is as usual: variables denote arbitrary subsets of such frames and the non-atomic sentences are evaluated as usual. We say that an \mathcal{L} -wff ϕ is *s-valid* iff it is valid in any separated interpretation on any separated frame. Clearly both $K + \text{NOM}_W$ and $K + \text{SWEEP}_W$ are sound with respect to this interpretation; everything provable from either basis is s-valid. However it is easy to falsify instances of both the NOM and SWEEP schemas. Let T be the frame $\langle \{-1, 0, 1\}, \{\langle 0, -1 \rangle, \langle 0, 1 \rangle\} \rangle$. Clearly -1 and 1 are a separated pair. Let V be any valuation that assigns $\{-1, 1\}$ to i , and $\{1\}$ to p . Then both an instance of NOM, $\Diamond(i \wedge p) \wedge \Diamond(i \wedge \neg p) \rightarrow \Diamond(i \wedge p \wedge \neg p)$, and an instance of SWEEP, $\Diamond(i \wedge p) \rightarrow \Box(i \rightarrow p)$, are false at 0 and thus cannot be derived from the weakened basis.

There are some simple observations we can make about the impact the addition of nominals has on the Henkin frame of the minimal normal modal logic. Suppose \mathcal{L} is a standard language of modal logic; that is, \mathcal{L} has a countably infinite set of variables and no nominals. Let K be the minimal normal logic in \mathcal{L} and \mathbf{H}^K its canonical frame. The following facts about \mathbf{H}^K are well known: \mathbf{H}^K is left directed, point generated, and indeed *strongly generated*. By this last is meant that there exists an $h \in H^K$ such that for all $h' \in H^K$, $h <_h h'$; from h we can get to any other point in one step. These properties follow from the fact that K admits the *Law of Disjunction* (LOD):

Definition 4.5.2 A modal logic L admits the Law of Disjunction (LOD) iff

$$\vdash_L \Box \phi_1 \vee \dots \vee \Box \phi_n \text{ implies } \vdash \phi_m, \text{ for some } m \text{ such that } 1 \leq m \leq n.$$

□

We briefly sketch why modal logics admitting LOD have the above properties.

Lemmon and Scott [61, Chapter 1, Section 3] showed that \mathbf{H}^L is left directed if L admits LOD as follows. Assume L admits LOD, and let $h, h' \in H^K$. Let

$$\Sigma = \{\Diamond \phi : \phi \in h\} \cup \{\Diamond \phi : \phi \in h'\}.$$

Σ is consistent; for if not: $\vdash_L \Diamond \phi_1 \wedge \dots \wedge \Diamond \phi_n \rightarrow \perp$, for some (finite) collection of wffs ϕ_m , ($1 \leq m \leq n$). But this means $\vdash_L \Box \neg \phi_1 \vee \dots \vee \Box \neg \phi_n$, and as L admits LOD, we must have that $\vdash_L \neg \phi_m$ for some m ($1 \leq m \leq n$). As ϕ_m is in one of the MCS h or h' , we have a contradiction. So Σ is consistent, and we can form Σ^∞ ; but by the (modal) Order Equivalence Lemma, Σ^∞ precedes both h and h' in the usual ordering on Henkin frames, which establishes left directedness.

Hughes and Cresswell [46, pages 96–100] showed that if L admits LOD, \mathbf{H}^L is strongly generated as follows. Define $\Sigma = \{\neg \Box \phi : \not\vdash_L \phi\}$. As in the Lemmon-Scott argument, Σ must be L -consistent on pain of contradiction, so again we can form Σ^∞ . But it is easy to see that for all $h' \in H^L$, $\Sigma^\infty <_h h'$; for if $\Box \phi \in \Sigma^\infty$, then by the construction of Σ , $\vdash_L \phi$, and L -theorems are in every L -MCS. This shows that \mathbf{H}^K is strongly generated, and hence point generated also.

Matters are different for languages with nominals.

Lemma 4.5.1 *Let \mathcal{L}' be any extension of \mathcal{L} with at least one nominal, i . Then the minimal modal logic in \mathcal{L}' does not admit LOD.*

Proof:

Clearly any \mathcal{L}' -instance of $\Diamond(i \wedge \phi) \wedge \Diamond(i \wedge \neg \phi)$ is K_{nm} -inconsistent. But such instances have the form $\neg \Box \neg(i \wedge \phi) \wedge \neg \Box \neg(i \wedge \neg \phi)$ and from the Hughes and Cresswell proof sketched above we know that if K_{nm} admits LOD then all sets $\{\neg \Box \phi : \vdash_{K_{nm}} \phi\}$ are consistent. As neither $i \wedge \phi$ nor $i \wedge \neg \phi$ is a K_{nm} theorem, K_{nm} cannot admit LOD. \square

In fact everything tumbles to ground: $H^{K_{nm}}$ cannot be left directed as no MCS h can precede both $\{i \wedge \phi\}^\infty$ and $\{i \wedge \neg \phi\}^\infty$; and as an immediate corollary of this we have that $H^{K_{nm}}$ cannot be strongly generated. In fact, it can't even be generated: for arbitrary existential modalities \Diamond^n and \Diamond^m , $\Diamond^n(i \wedge \phi) \wedge \Diamond^m(i \wedge \neg \phi)$ is inconsistent, and thus no MCS h can precede both $\{i \wedge \phi\}^\infty$ and $\{i \wedge \neg \phi\}^\infty$, no matter how many steps intervene.

The only obvious thing we can say about the structure of $H^{K_{nm}}$ derives from the following observation: the following special case of LOD is unaffected by the addition of nominals: $\vdash_{K_{nm}} \phi$ iff $\vdash_{K_{nm}} \Box \phi$. The 'only if' direction is immediate by necessitation; the less trivial 'if' direction follows because given a falsifying NTL model for ϕ , we can 'root' the model by adjoining a new point that precedes all the other points in the model, but that is not itself preceded by any of the others. Given that ϕ is falsifiable in the original model, evaluating $\Box \phi$ at this new point yields -1 , and we have shown the contrapositive of the desired result. As a corollary we have that $H^{K_{nm}}$ is left unbounded, for this 'rooting' argument shows that for all $h \in H^{K_{nm}}$, $\{\Diamond \phi : \phi \in h\}$ is consistent, and thus we can always find a predecessor of any point in the Henkin frame. So some asymmetry remains, but it is hard to say anything more about the structure of $H^{K_{nm}}$; it is not even clear whether it is connected.

Axiomatising the minimal logic for multimodal languages is straightforward. Suppose Δ is the modality index for some multimodal language \mathcal{L}_Δ . Then by an \mathcal{L}_Δ existential modality is meant the smallest set ET_Δ containing: the null sequence; \Diamond_δ , for all $\delta \in \Delta$; and $\Diamond_\delta E$, if $E \in ET_\Delta$, for all $\delta \in \Delta$. Universal modalities are defined analogously.

Again with E and A read in this way, either NOM or SWEEP can be added to the usual axiomatisation of the minimal multimodal logic ⁷ to yield a complete axiomatisation of the \mathcal{L} validities. ⁸ To show this using the double generation technique we generate simultaneously all the relations $<_\delta$. That is, at each stage we take the smallest subset of the Henkin multiframe that is closed under all the relations.

4.6 The GPT axiomatisation

Let us now consider the Gargov, Passy and Tinchev axiomatisation of the minimal modal logic for languages with nominals. They first define necessity and possibility forms:

Definition 4.6.1 *Let \mathcal{L} be a language of NTL, $\$$ be a new entity distinct from any \mathcal{L} wff or symbol, and θ be a wff of \mathcal{L} . Then the necessity forms of \mathcal{L} , are the elements of the smallest set \Box -form such that:*

$$\begin{aligned} \$ &\in \Box\text{-form} \\ L \in \Box\text{-form} &\text{ implies } \theta \rightarrow L \in \Box\text{-form} \\ L \in \Box\text{-form} &\text{ implies } \Box L \in \Box\text{-form}; \end{aligned}$$

and the possibility forms of \mathcal{L} , are the elements of the smallest set \Diamond -form such that:

$$\begin{aligned} \$ &\in \Diamond\text{-form} \\ L \in \Diamond\text{-form} &\text{ implies } \theta \wedge L \in \Diamond\text{-form} \\ L \in \Diamond\text{-form} &\text{ implies } \Diamond L \in \Diamond\text{-form}. \end{aligned}$$

⁷Simply ' K with indices'. That is, for all $\delta \in \Delta$, we have the axiom schema: $\Box_\delta(\phi \rightarrow \psi) \rightarrow (\Box_\delta\phi \rightarrow \Box_\delta\psi)$, and the rules: $\vdash \phi \Rightarrow \vdash \Box_\delta\phi$. (As usual we have the PC axiom schema and modus ponens.)

⁸The orderings needed for the \mathcal{L}_Δ Henkin multiframe are defined by $h <_\delta h'$ iff $\Box_\delta\phi \in h$ implies $\phi \in h'$, for all wffs ϕ . Thus the \mathcal{L}_Δ Henkin multiframe is just $\langle H, \{<_\delta\}_{\delta \in \Delta} \rangle$, where H is the set of all MCS in language \mathcal{L}_Δ .

If ψ is any wff of \mathcal{L} , and L and M are \Box -forms and \Diamond -forms respectively, then by $L(\psi)$ and $M(\psi)$ are meant the \mathcal{L} -wffs obtained by replacing the (unique) occurrence of $\$$ in L and M respectively by ψ . \square

They then axiomatise the minimal logic for languages of weak NML by adding to the usual axioms of the minimal modal logic K all instances of the following schema:

$$\mathbf{Ax}_N \quad M(n \wedge \phi) \rightarrow L(n \rightarrow \phi),$$

where L and M are metavariables over \Box -forms and \Diamond -forms respectively. They prove completeness by a three point argument on generated subframes of the $\mathbf{H}^{K_{nm}}$.

The form of the Ax_N schema is superficially reminiscent of that of SWEEP, but the M and the L don't range over universal and existential modalities but over the more complex \Box - and \Diamond -forms. Thus for fixed i and ϕ the consequents of Ax_N include all entries in the following infinite matrix:

$(i \rightarrow \phi)$	$\Box(i \rightarrow \phi)$	$\Box\Box(i \rightarrow \phi)$	\dots
$\phi \rightarrow (i \rightarrow \phi)$	$\Box(\phi \rightarrow (i \rightarrow \phi))$	$\Box\Box(\phi \rightarrow (i \rightarrow \phi))$	\dots
$\phi \rightarrow (\phi \rightarrow (i \rightarrow \phi))$	$\Box(\phi \rightarrow (\phi \rightarrow (i \rightarrow \phi)))$	$\Box\Box(\phi \rightarrow (\phi \rightarrow (i \rightarrow \phi)))$	\dots
\vdots	\vdots	\vdots	\vdots
$\phi \rightarrow \Box(i \rightarrow \phi)$	$\Box(\phi \rightarrow \Box(i \rightarrow \phi))$	$\Box\Box(\phi \rightarrow \Box(i \rightarrow \phi))$	\dots
\vdots	\vdots	\vdots	\vdots

The antecedents of Ax_N , again for fixed i and ϕ , consist of all entries in the matrix obtained from that above by replacing \Box by \Diamond and \rightarrow by \wedge . Note that for fixed i and ϕ the SWEEP schema consists merely of conditionals formed from the first row of each of the above matrices. The simpler SWEEP_w schema that suffices for tense logic essentially consists, for fixed i and ϕ , of only the single wff occurring in the top left entry of the second matrix — $i \wedge \phi$ — as antecedent; and as consequents just the wffs in the first row of the above matrix. Thinking in terms of paths and path equations is a simpler way of adding nominals to modal (and especially tensed) languages.

The authors then note that Ax_N suffices to axiomatise ordinary and strong languages as well: for the former they suggest adding ^asingle point to ‘totalise’ the model. ⁹ They prove the axiomatisation is complete for strong languages by the filtration argument we used in Chapter 1. Actually they do more: they also augment the logic by an infinitary rule of inference, COV , and show that adding COV as a rule yields a completeness result for strong languages; the filtration argument is then used to show that COV is eliminable. But, as they comment, COV need not be eliminable in all extensions of K_{nm} . The COV rule is very interesting and we discuss it at the end of the next chapter.

⁹See my ‘point at infinity’ comment near the end of the Makinson proof.

Chapter 5

Extensions of K_{nt}

In this chapter we axiomatise the logics of some interesting classes of frames and establish their decidability.

Languages of NTL inherit a number of completeness results straightforwardly from Priorean tense logic. The following table lists some useful schemata together with the effect their inclusion as axioms has on Henkin frames: ¹

4	$FF\phi \rightarrow \phi$	Transitivity
T	$\phi \rightarrow F\phi$	Reflexivity
D_R	$F\top$	Right unboundedness
D_L	$P\top$	Left unboundedness
D	$P\top \wedge F\top$	Unboundedness

Thus the nominal tense logic of transitive frames is axiomatised by adding to K_{nt} all instances of the schema 4 — which we would write as $K_{nt} + 4$ or $K_{nt}4$ — and the nominal tense logic of unbounded frames is axiomatised by $K_{nt}D$. Proving this is simple: first we doubly generate a model from the relevant Henkin frame in the usual way. As before, the NOM_W schema included as part of our base logic K_{nt} guarantees that the Unique Occurrence Lemma holds. Further, the additional structural axioms force the

¹The results for the D variants are immediate. That 4 and T have the desired effect is clear if their dual forms $G\phi \rightarrow GG\phi$ and $G\phi \rightarrow \phi$ are considered.

whole Henkin frame, and thus our generated subframes, to have the relational property in question. Hence, by the Strong Completeness Lemma, we have completeness. Note that the above results are additive: for example, K_n4T is complete with respect to the class of preorders.

This is useful, but unsurprising. Moreover these completeness theorems deal only with classes of frames already definable in Priorean languages. How do we axiomatise the newly definable classes, and can the defining formulas be used as axioms? They can, but not in so straightforward a fashion as for the examples listed above. In particular, although any instance of $n \rightarrow \neg Fn$ defines irreflexivity, the inclusion of all instances of this schema as axioms does *not* guarantee an irreflexive Henkin frame; and although any instance of $n \rightarrow G(Fn \rightarrow n)$ defines antisymmetry, the inclusion of all instances of this schema as axioms does not guarantee an antisymmetric Henkin frame. These problems form the starting point for the chapter.

In the first section we examine the simplest problematic logic, the logic of irreflexive frames. We prove a simple lemma, the Irreflexivity lemma, which tells us that although the natural axiom schema does not force the Henkin frame to be irreflexive, it does force all points containing nominals to be irreflexive. We then show how our Henkin model can be ‘structurally rectified’ — in this case, turned into an irreflexive model — in a way that does not affect the unique occurrence property. This discussion illustrates in very simple form the strategy that underlies much of the chapter.

In the following section we give a detailed account of a powerful rectification technique that will be used throughout the thesis: Segerberg’s bulldozing method. We first use this to axiomatise the logic of the SPOs, and then, in the following section, to axiomatise the POs.

At last we are ready to examine the logics of linear frames, and at this stage a number of interesting things happen. Firstly, we no longer need to double generate our models; a single generative step suffices. Secondly, finite axiomatisations become possible on these frames. Thirdly, we note a peculiarity of the nominal modal logics of linear frames.

By this stage we have an interesting crop of logics; next we consider the question

of their decidability. In spite of an apparent obstacle — many of the interesting logics lack the finite frame property — we *can* use simple filtration arguments to establish the decidability of all the logics introduced, and we do so.

In the final section we axiomatise the logics of \mathbf{Q} , \mathbf{R} , \mathbf{Z} and \mathbf{N} .

As has already been indicated, this chapter makes heavy use of the classic Segerberg method of cluster analysis. In fact, the chapter could be read as a rather lengthy footnote to Segerberg's "Modal Logics with linear alternative relations" [97], indicating how the techniques introduced there adapt to languages with nominals. For the most part I have used Segerberg's original notation and terminology, though in certain places I have used ideas from Robert Goldblatt's more recent [40, Chapter 8], which contains an original and very clear account of cluster analysis.

5.1 The logic of irreflexive frames

The axiomatisation I is obtained by adding all instances of

$$n \rightarrow \neg Fn \quad (\mathbf{I})$$

to K_{nt} . (As usual n is a metavariable over nominals.) Note that if we are working with a purely Priorean language, I collapses into K_t as there are no instances of \mathbf{I} or \mathbf{NOM}_W .

I is characterised by the class of irreflexive frames. Soundness is immediate: we need only check that all instances of \mathbf{I} are valid on the class of irreflexive frames — but any instance defines that class. What about completeness?

We must show that any I -consistent set of sentences Σ has a model whose underlying frame is irreflexive. Given such a set Σ , doubly generate a Henkin model $\langle \mathbf{T}, V_t \rangle$ in the usual fashion. Here we run into a problem. Certainly this model verifies all the sentences in Σ , but there is no guarantee that \mathbf{T} is irreflexive: there may be points of this frame containing no nominals, and we cannot guarantee the irreflexivity of such points. Nonetheless, we do have the following:

Lemma 5.1.1 (Irreflexivity Lemma) *Let t be any element of T containing a nominal. Then $t \not<_t t$.*

Proof:

Suppose t contains a nominal, say i , and $t <_t t$. As $i \rightarrow \neg Fi$ is an axiom of I , $\neg Fi \in t$. That is, $G\neg i \in t$. But $t <_t t$ means that $G\phi \in t$ implies $\phi \in t$, and thus we have $\neg i \in t$. As t is an MCS, this is a contradiction. \square

If we could find some means of ‘structurally rectifying’ our generated model — that is, if we could find a means of turning it into an irreflexive model that verified the same formulas — we would have our result. Of course we must take care to choose a rectification procedure that does not destroy the unique occurrence property enjoyed by the original model. Such a well behaved process for I is easy to describe: if we ‘stretched apart’ each reflexive point t in \mathbf{T} into two points s and s' , each of which preceded the other, and neither of which preceded itself, we would have an equivalent model as each point would have access to exactly the same information as before. By the Irreflexivity Lemma this process would not affect the unique occurrence property: points containing nominals are irreflexive, and thus are never stretched. More precisely, given \mathbf{T} define a new stretched frame \mathbf{S} as follows:

$$S = \{\langle t, 0 \rangle : t \in T \text{ and } t \not<_t t\} \cup \{\langle t, 1 \rangle, \langle t, 2 \rangle : t \in T \text{ and } t <_t t\},$$

and if $\langle t, m \rangle, \langle t', n \rangle \in S$, ($0 \leq m, n \leq 2$), then:

$$\langle t, m \rangle <_s \langle t', n \rangle \text{ iff } t <_t t', \text{ and } t = t' \text{ implies } m \neq n.$$

Clearly $<_s$ is an irreflexive relation. Note that points t of T containing nominals can appear only in pairs of the form $\langle t, 0 \rangle$ in \mathbf{S} .

Now define a valuation V_s on \mathbf{S} by $V_s(a) = \{\langle t, m \rangle \in S : t \in V_t(a)\}$, for all atoms a . This is a valuation: as $V_t(i)$ is some singleton subset $\{t_i\}$ of T , $V_s(i)$ is the singleton subset $\{\langle t_i, 0 \rangle\}$ of S .

All that remains is to check that $\langle \mathbf{T}, V_t \rangle$ and $\langle \mathbf{S}, V_s \rangle$ are equivalent models. Consider the function $f : S \rightarrow T$ defined by $f(\langle t, m \rangle) = t$. It is a p-morphism from \mathbf{S} to \mathbf{T} , and

moreover $a \in V_s(a)$ iff $f(a) \in V_t(a)$, for all atoms a . Thus $\langle T, V_t \rangle$ is a p-morphic image of $\langle S, V_s \rangle$ and the two models are equivalent. Thus every I -consistent set of sentences can be verified on an irreflexive model, establishing:

Theorem 5.1.1 *I is strongly complete with respect to the class of irreflexive frames.* \square

I is a rather simple axiomatisation, and the class of all irreflexive frames is not of much interest as far as temporal modeling is concerned; nonetheless, the above completeness proof is useful as it shows clearly the form of the arguments that will be encountered in the rest of this chapter. In general, the Henkin model produced by the double generation process cannot be guaranteed to have the desired relational structure and some form of rectification is required. We usually proceed by finding a method of producing a model $\langle S, V_s \rangle$ of which our generated Henkin model $\langle T, V_t \rangle$ is a p-morphic image — exactly what was done above. The only difference is one of complexity: the ‘stretching’ process is simple and obvious, whereas for the more complex logics to follow we need more powerful rectification techniques. Another point which recurs in later proofs is the way in which points containing nominals *could* be guaranteed to have the appropriate relational behaviour even though an arbitrary point couldn’t. This enabled such points to be left alone: our rectification technique only had to be applied to points not containing nominals. Rectification was local.

As the above completeness proof works for any language of NTL, it shows that the minimal tense logic for standard languages, K_t , is complete not only with respect to the class of all frames, but with respect to the irreflexive frames also. This result is well known and is perhaps the simplest of the ‘sharpening’ results mentioned in the last chapter. This sharpening is due to the inability of purely Priorean languages to talk about irreflexivity: there simply are no validities in Priorean languages peculiar to irreflexive frames, and hence nothing new for the minimal logic to prove. That is, the positive sharpening result — the minimal axiomatisation K_t suffices even for the irreflexive frames — is a reflection of a negative expressibility result.

As all instances of $n \rightarrow \neg Pn$ are valid on the class of irreflexive frames, by the above completeness theorem all instances are I -provable. I -proofs are easily displayed:

1.	i	Assumption	
2.	$i \rightarrow HFi$	T4	1
3.	HFi	1,2; MP	1
4.	$Fi \rightarrow \neg i$	I contraposed	1
5.	$H(Fi \rightarrow \neg i)$	4; HGen	1
6.	$HFi \rightarrow H\neg i$	5; T2, MP	1
7.	$i \rightarrow H\neg i$	2,6; PC	1
8.	$H\neg i$	1,7; MP	1
9.	$\neg Pi$	8	1
10.	$i \rightarrow \neg Pi$	1,9; Discharge	

Adding all instances of $n \rightarrow \neg \Diamond n$ to K_{nm} axiomatises the nominal modal logic of the irreflexive frames; the argument given above establishes this.

5.2 The logic of SPOs

$I4$ is the axiomatisation obtained by adjoining all instances of the following two schema:

$$n \rightarrow \neg Fn \quad (I)$$

$$FF\phi \rightarrow F\phi \quad (4)$$

to the axioms of K_{nt} . (As usual, n is a metavariable over nominals and ϕ a metavariable over arbitrary wffs.) Note that all I -theorems are $I4$ -theorems. The inclusion is proper as no instance of 4 is I -provable by the soundness result for I . If we are working with a purely Priorean language $I4$ collapses into K_4 .

$I4$ is an important axiomatisation as it is characterised by the class of SPOed frames, and thus captures what can reasonably be considered the minimal temporal logic in languages of NTL. $I4$ is obviously sound with respect to these frames, but as should be clear from the discussion of I , establishing completeness will be rather tricky. Let's

examine the problem. As usual, given a consistent set of sentences Σ , we doubly generate a Henkin frame \mathbf{T} and then form a Henkin model by defining the natural valuation V_t . Because of the presence of the instances of schema 4, \mathbf{T} will be transitive; but as we have already seen, the presence of the instances of schema I does not guarantee irreflexivity — only the irreflexivity of points containing nominals is assured. So how do we turn $\langle \mathbf{T}, V_t \rangle$ into an equivalent model that is transitive and irreflexive? The simple stretching procedure used for I will not do: stretching a reflexive point t apart into two points s and s' certainly yields an irreflexive frame, but destroys transitivity in so doing, as now $s <_t s'$ and $s' <_t s$, but $s \not<_t s$. Indeed a little experimentation should demonstrate that the problem of constructing an irreflexive and transitive model equivalent to $\langle \mathbf{T}, V_t \rangle$ is non-trivial. Fortunately there is a technique due to Krister Segerberg which can be applied: bulldozing. An account of how to irreflexively bulldoze a transitive frame $\langle \mathbf{T}, <_t \rangle$ will now be given.

As with the stretching argument we are going to perform surgery ² on \mathbf{T} . We're going to snip out the troublesome parts of the generated frame and sew in something more to our liking. But what are the troublesome parts of the frame? The obvious answer is: 'any reflexive points'; but while this is correct, it is at the wrong level of abstraction. A better answer is: any *cluster*. (Remember from Chapter 2 that a cluster is a maximal equivalence relation on \mathbf{T} .) The point is this: we shouldn't think in terms of removing single reflexive points, but of removing whole clusters at a time. We need to construct a frame equivalent to \mathbf{T} , and by removing clusters as a single lump we've kept together information that, intuitively, needs to be kept together. 'Thinking in terms of clusters' is the first important intuition behind bulldozing.

But to make an equivalent model, what do we replace the clusters with? To gain our desired completeness theorem it must be some sort of SPO — and in fact certain STOs will be used — but how? Segerberg's answer is elegant. Note that any cluster, even a

²This term is taken from Goldblatt [40, Page 54]. As will become apparent, the surgery that patient \mathbf{T} requires is a transplant.

simple one, introduces an infinity of information recurrence. Given any cluster C we can loop within C arbitrarily many times, visiting each point and its associated information as often as we please — clusters contain arbitrarily long paths. Thus when we replace C (somehow) by a STO we must take care that this STO duplicates all the information in C infinitely often. What the bulldozing procedure does is to impose an arbitrary strict total order on the information in a cluster — that is, it picks some route through the cluster that visits each point once and once only — and then lays out infinitely many copies of this path in both the forward and backwards direction.³ The final step is to replace each cluster C by this infinite repetition of the chosen path through C : the fat clusters of \mathbf{T} have been squashed into thin and infinitely long STOs: hence ‘bulldozing’.

We now give the full definition. Let the clusters of \mathbf{T} be indexed by some suitable set Δ . We assume that for each $\delta \in \Delta$, C_δ is embedded in a set C_δ^\dagger in such a way that if C_θ and C_λ are distinct clusters of \mathbf{T} then $C_\theta^\dagger \cap C_\lambda^\dagger = \emptyset$. (This is a refinement of the basic bulldozing idea that will be important later — we introduce it here even though for the simpler axiomatisations such as *I4* it is unnecessary. For simple axiomatisations we merely choose the identity embedding of each cluster into itself.) For each $\delta \in \Delta$ choose an arbitrary $c \in C_\delta$; we term this arbitrary element $\gamma(C)$. (That is, $\gamma : \{C_\delta : \delta \in \Delta\} \rightarrow \bigcup_{\delta \in \Delta} C_\delta$ is a choice function.) Next, for each $\delta \in \Delta$ we define an arbitrary strict total order $<_\delta$ on C_δ . This marks an important choice point in bulldozing. What we are doing here is *irreflexive* or *heavy* bulldozing. Our objective is to grind down *all* the clusters, including the simple ones, as we want an irreflexive frame — hence the arbitrary order we impose on C_δ is a STO. In the next section when we consider the logic of POs we will want a reflexive frame — so we will leave the simple clusters untouched, bulldoze just the proper clusters, and at this point of the construction will define an arbitrary TO on C_δ ; that is, a transitive, trichotomous, antisymmetric and *reflexive* relation. This variant will be described as *reflexive* or *light bulldozing*.

³Slightly more formally, once we have chosen our arbitrary STOed route R through the cluster, we pick a second arbitrary unbounded STO (say Z), and form the lexicographical product $Z \odot R$.

Finally, let $\alpha = \langle A, <_A \rangle$ be any order type without first or last element — for example the order type of \mathbf{Q} or \mathbf{Z} . Define C_δ^* to be $C_\delta^\dagger \times A$. All the pieces are now to hand, and we can describe the set underlying our new frame. Define:

$$S = (T \setminus \bigcup_{\delta \in \Delta} C_\delta) \cup \bigcup_{\delta \in \Delta} C_\delta^*.$$

That is, we've removed all the clusters from the frame — the things being bulldozed flat — and have sewn in place our new crossproducts C_δ^* , the results of the squashing.⁴ There is a potential difficulty here: what if $T \cap \bigcup_{\delta \in \Delta} C_\delta^* \neq \emptyset$? However it should be clear that this is only a matter of bookkeeping. Trouble can only arise if we are careless as to the sets C_δ^\dagger that the clusters are embedded in; clearly judicious set-theoretic juggling — indexing, pair formation etc. — will remove such pseudo-problems, and thus we can assume without loss of generality that these two sets are disjoint.

Define a function $f : S \rightarrow T$ by:

$$f(s) = \begin{cases} s & \text{if } s \in T \\ c & \text{if } s = \langle c, a \rangle \text{ and } c \in T \\ \gamma(C_\delta) & \text{if } s = \langle c, a \rangle, c \notin T, \text{ and } c \in C_\delta \end{cases}$$

Suppose $s_1, s_2 \in S$. Define an ordering $<_s$ on S by $s_1 <_s s_2$ iff:

1. $s_1 \in T$ or $s_2 \in T$ and $f(s_1) <_t f(s_2)$; or
2. $s_1 = \langle c_1, a_1 \rangle$ and $s_2 = \langle c_2, a_2 \rangle$ and:

either there are distinct clusters C_1 and C_2 such that $c_1 \in C_1^\dagger$, $c_2 \in C_2^\dagger$ and $\gamma(C_1) <_t \gamma(C_2)$,

or there is a cluster C_δ such that $c_1, c_2 \in C_\delta^\dagger$ and $a_1 <_A a_2$,

⁴Strictly speaking we can't yet say that our crossproducts C_δ^* are 'in place' as we have not yet defined the ordering on S ; and for the same reason we can't yet see why the C_δ^* are 'flat'. Both matters will shortly be attended to.

or there is a cluster C_δ such that $c_1, c_2 \in C_\delta^\dagger$, and $a_1 = a_2$, and $c_1 <_\delta c_2$.

Lemma 5.2.1 S is a SPOed frame.

Proof:

Case by case examination. □

Lemma 5.2.2 f is a p -morphism from S onto T .

Proof:

f is clearly surjective. To check that f forward preserves $<_\delta$, suppose that $s_1, s_2 \in S$ and $s_1 <_\delta s_2$; we need to show that $f(s_1) <_\delta f(s_2)$. If either $s_1 \in T$ or $s_2 \in T$ this is immediate, so suppose $s_1 = \langle c_1, a_1 \rangle$ and $s_2 = \langle c_2, a_2 \rangle$. If s_1 and s_2 belong to distinct crossproducts C_1^* and C_2^* then as $s_1 <_\delta s_2$ we have $\gamma(C_1) <_\delta \gamma(C_2)$. But $f(s_1) \in C_1$ and $f(s_2) \in C_2$, and as $<_\delta$ is an equivalence relation on any cluster we have:

$$f(s_1) <_\delta \gamma(C_1) <_\delta \gamma(C_2) <_\delta f(s_2)$$

and hence $f(s_1) <_\delta f(s_2)$ by the transitivity of $<_\delta$. On the other hand if both s_1 and s_2 belong to the same crossproduct C_δ^* then $f(s_1)$ and $f(s_2)$ are both in the same cluster C_δ and hence $f(s_1) <_\delta f(s_2)$.

Now we must check that f partially backwards preserves $<_\delta$. Assume $s_1, s_2 \in S$ and $f(s_1) <_\delta f(s_2)$; we need to show that there is an $s_3 \in S$ such that $f(s_3) = f(s_2)$ and $s_1 <_\delta s_3$. Now if either s_1 or $s_2 \in T$ then by the first defining clause for $<_\delta$, $s_1 <_\delta s_2$ and so the choice $s_3 = s_2$ suffices. So suppose neither s_1 nor s_2 are in T ; that is, $s_1 = \langle c_1, a_1 \rangle$ and $s_2 = \langle c_2, a_2 \rangle$. There are two subcases to consider.

Firstly suppose $c_1 \in C_1^\dagger$ and $c_2 \in C_2^\dagger$ where C_1 and C_2 are distinct clusters. Then as $f(s_1) \in C_1$, $f(s_2) \in C_2$, and $f(s_1) <_\delta f(s_2)$ we have, because of the equivalence property of clusters, that:

$$\gamma(C_1) <_\delta f(s_1) <_\delta f(s_2) <_\delta \gamma(C_2)$$

and thus $\gamma(C_1) <_t \gamma(C_2)$ by $<_t$ transitivity — but this guarantees that $s_1 <_s s_2$ and so again we choose $s_3 = s_2$.

Secondly suppose there is a cluster C_δ such that $c_1, c_2 \in C_\delta^\dagger$. As α is a STO it is trichotomous. If $a_1 < a_2$ choose $s_3 = s_2$. If $a_1 = a_2$ or $a_2 <_A a_1$ then choose an $a_3 \in A$ such that $a_1 <_A a_3$; such an a_3 exists as α has no last element. Let $s_3 = \langle c_2, a_3 \rangle$. This choice gives $s_1 <_s s_3$ and $f(s_1) = f(s_3)$ as required.

The other part of the backwards clause for (temporal) p-morphisms — that if $f(s_1) <_t f(s_2)$ then there is an s_3 such that $f(s_3) = f(s_1)$ and $s_3 <_s s_2$ — is established analogously; note that in the last part we will appeal to the fact that α has no *first* element. So \mathbf{T} is a p-morphic image of \mathbf{S} under f . \square

We now have everything we need to establish our completeness theorem:

Theorem 5.2.1 *I4 is strongly complete with respect to the class of SPO frames.*

Proof:

Doubly generate the Henkin model $\langle \mathbf{T}, V_t \rangle$ for an *I4*-consistent set of sentences Σ in the usual way, and heavily bulldoze \mathbf{T} , embedding each cluster identically into itself, and choosing (say) \mathbf{Z} for α . This creates the SPOed frame \mathbf{S} . Let f be the p-morphism from \mathbf{S} to \mathbf{T} described above. Define a function $V_s : \text{ATOM}_\mathcal{L} \rightarrow \text{Pow}(S)$ by $s \in V_s(a)$ iff $f(s) \in V_t(a)$ for all atoms a . It is easy to see that V_s is a valuation — all we need to check is that all the nominals are assigned singletons. But the Irreflexivity Lemma tells us that every point t of \mathbf{T} containing a nominal is irreflexive and hence in no cluster, and thus no such point is bulldozed. (Intuitively, the bulldozing process does not duplicate nominals.) More precisely, this means that if t contains a nominal then $f^{-1}(t)$ is a singleton subset of S , as inspection of the definition of f shows. Hence V_s is a valuation; and clearly $\langle \mathbf{T}, V_t \rangle$ and $\langle \mathbf{S}, V_s \rangle$ are p-morphic models and thus equivalent. Hence Σ is verified on an SPO model and we have our strong completeness result. \square

Note that once again a classic sharpening result has been lost for languages with nominals: the above completeness proof works for any language of NTL and thus shows

that for purely Priorean languages not only is K_4 complete with respect to the transitive frames, but with respect to the SPOs also. Clearly the analogous sharpening for languages with nominals is lost: $K_{nt}4$ is *not* complete with respect to the SPOs as (by soundness) no instance of I is provable.

The SPOs are the most geometrically interesting class of frames we have axiomatised so far, and this is reflected in the theorems of $I4$. Firstly note that all SPO's are antisymmetric, so we expect every instance of $n \rightarrow G(Fn \rightarrow n)$ to be $I4$ -provable as any instance defines antisymmetry. By completeness such $I4$ -proofs exist; here is one:

1.	$\neg G(Fi \rightarrow i)$	Assumption	
2.	$F\neg(Fi \rightarrow i)$	1	1
3.	$F(Fi \wedge \neg i)$	2; PC	1
4.	FFi	3; TL	1
5.	$FFi \rightarrow Fi$	4	1
6.	Fi	4, 5; MP	1
7.	$Fi \rightarrow \neg i$	I contraposed	1
8.	$\neg i$	6, 7; MP	1
9.	$\neg G(Fi \rightarrow i) \rightarrow \neg i$	1, 8; Discharge	
10.	$i \rightarrow G(Fi \rightarrow i)$	9, Contraposition	

(Note that the annotation on line 5 of the above proof means that it is an instance of schema 4, not that it was obtained from line 4 of the proof.) The proof that $\vdash_{I4} i \rightarrow H(Pi \rightarrow i)$ is obtained analogously. ⁵

⁵In giving a syntactic proof of this mirror image formula we need to use the fact that $\vdash_{I4} PPi \rightarrow Pi$. By the above completeness theorem such a proof exists, but finding one is tricky. Proofs can be found in [5, page 178], or [63, page 26].

SPOs are also asymmetric, thus we expect all instances of $n \rightarrow \neg FFn$ to be theorems.

This is easy:

1.	i	Assumption	
2.	$i \rightarrow \neg Fi$	I	1
3.	$\neg Fi$	1, 2; MP	1
4.	$\neg Fi \rightarrow \neg FFi$	4 Contraposed	1
5.	$\neg FFi$	3, 4; MP	1
6.	$i \rightarrow \neg FFi$	1, 5; Discharge	

In fact we can say more than this.

Definition 5.2.1 A frame \mathbf{T} is said to be m -asymmetric iff whenever there are m points of T such that $t_1 < \dots < t_m$ then $t_m \not\prec t_1$. \square

For any $m \geq 1$ we can define the class of m -asymmetric frames in languages with nominals. 1-asymmetry is just irreflexivity, defined by any instance of $n \rightarrow \neg Fn$. 2-asymmetry is just asymmetry simpliciter, defined by any instance of $n \rightarrow \neg FFn$. In general we have that for all $m \geq 1$ that m -asymmetry is defined by any instance of $n \rightarrow \neg F^m n$, where F^m (as usual) denotes a sequence of F 's of length m . Clearly any instance of $n \rightarrow \neg P^m n$ also defines the same class.⁶

Now if \mathbf{T} is a SPO we have that for all $m \geq 1$, \mathbf{T} is m -asymmetric: transitivity allows all such m -length paths to be collapsed to paths of length 1, giving the 'no return' clause on pain of forming a reflexive loop. So by completeness all instances of $n \rightarrow \neg F^m n$ and $n \rightarrow \neg P^m n$ are theorems of $I4$. Exhibiting syntactic proofs is easy. All instances of I are axioms, so the base case for an inductive construction is given. So suppose that for some $m \geq 1$ we have I_{n4} -proofs of all instances of $n \rightarrow \neg F^m n$. Then it is easy to modify the above proof that $\vdash_{I4} i \rightarrow \neg FFi$ to yield a proof that $\vdash_{I4} i \rightarrow \neg F^{m+1}i$: simply uniformly

⁶Note that for all $m \geq 1$, m -asymmetric frames are not definable in purely Priorean languages: \mathbf{Z} is m -asymmetric for all such m , and for no $m \geq 0$ does $\langle 0, \{0, 0\} \rangle$ have this property, and the familiar p-morphic collapse of the former to the latter destroys all hopes.

substitute $\neg F^m i$ for $\neg Fi$, and $\neg F^{m+1} i$ for $\neg FFi$. The new line 2 is justified by our inductive hypothesis, while the new line 4 is justified by noting that from $FFi \rightarrow Fi$ we can obtain $F^{m+1} \rightarrow F^m i$ by $m - 1$ applications of GGen and modus ponens, and hence we have an inductive specification of how to construct a proof of any desired instance. The proofs of instances of $n \rightarrow \neg P^m n$ are constructed analogously, appealing to the fact that $\vdash_{I4} PPi \rightarrow Pi$.

$I4$ underlies many of the logics of linear frames we later consider: here is a simple non-linear extension. The logic of unbounded SPOs is axiomatised by $I4D$. The proof is simply that the inclusion of the instances of D forces the doubly generated Henkin model to be unbounded, and unboundedness clearly is not destroyed by bulldozing — if anything, it is accentuated by it!

The modal logic of the SPOs can be axiomatised by adding all instances of

$$\Box \rightarrow \neg \Diamond \Box \quad (I)$$

$$\Diamond \Diamond \phi \rightarrow \Diamond \phi \quad (4)$$

to the axioms of K_{nm} . Soundness is immediate, and completeness again follows by a bulldozing argument. The proof given above works; but note that we need not require our choice of α to be an order type unbounded in both directions, merely that it be right unbounded — thus \mathbb{N} could be used in place of \mathbb{Z} . This is yet another reflection of the fact that with modal languages we can only look forwards, never back.

5.3 The logic of POs

PO is the axiomatisation obtained by adding as axioms all instances of

$$\phi \rightarrow F\phi \quad (T)$$

$$FF\phi \rightarrow F\phi \quad (4)$$

$$n \rightarrow G(Fn \rightarrow n) \quad (Anti)$$

to the axioms of K_{nt} . When working with purely Priorean languages this collapses into S_4 , the standard tense logical axiomatisation of the preorders.

PO is characterised by the class of POs. Soundness is obvious; completeness can be shown by means of a bulldozing argument as follows. Once more suppose that given our PO -consistent set of sentences Σ we have doubly generated a Henkin model $\langle \mathbf{T}, V_i \rangle$. This model is guaranteed to be both reflexive and transitive due to the presence of all instances of 4 and T in every $t \in T$, but antisymmetry is not assured. We do, however, have the following:

Lemma 5.3.1 (Simple Cluster Lemma) *Let t be any element of T containing a nominal. Then $t <_t t'$ and $t' <_t t$ implies $t = t'$.*

Proof:

Suppose t contains a nominal, say i , and that both $t < t'$ and $t' < t$. Then $G(Fi \rightarrow i) \in t$, and as $t <_t t'$, $Fi \rightarrow i \in t'$. But as $t' < t$, $Fi \in t'$ and hence $i \in t'$. But \mathbf{T} has the unique occurrence property and thus $t = t'$. \square

Now because \mathbf{T} is reflexive, every point $t \in T$ is in a cluster; but what the previous lemma establishes is that *no point containing a nominal is in a proper cluster*. Points containing nominals are simple clusters. Our strategy is clear: we must lightly bulldoze \mathbf{T} , and bulldoze only proper clusters. ⁷

We sketch the changes that need to be made to the bulldozing construction outlined earlier. There are only two. Firstly, we only index all *proper* clusters of \mathbf{T} . Thus in all the definitions involving C_δ , C_δ^\dagger and C_δ^* in the bulldozing construction, everything is relativised to the proper clusters — and, in particular, only proper clusters are removed

⁷The construction sketched below differs slightly from Segerberg's account of light bulldozing in this last mentioned detail. Even when lightly bulldozing, Segerberg bulldozes *all* clusters, including the simple ones. When dealing with standard languages this is harmless; for languages with nominals, on the other hand, such bulldozing would be disastrous: each nominal would be smeared across an infinite number of points, emphatically destroying the unique occurrence property.

from \mathbf{T} and replaced by the C_δ^* . Secondly, as was mentioned in the exposition of bulldozing, when choosing α we choose a TO instead of a STO, thus ensuring the reflexivity of all points in the C_δ^* , and hence of all points in \mathbf{S} . We thus guarantee that \mathbf{S} is a partial order and that the f of the bulldozing definition is a surjective p-morphism from \mathbf{S} to \mathbf{T} . The proof of both points is simple; indeed, because of the relativisation of the indexing to proper clusters the earlier given proof that f is a p-morphism is unchanged. We refer to this construction as *light bulldozing*. V_s is defined as in the previous completeness proof and again is a valuation, so once more we have an equivalent model of the desired form and thus:

Theorem 5.3.1 *PO is strongly complete with respect to the class of all partial orders.*

□

Once again this proof shows that the obvious sharpening result is lost for languages with nominals: S_4 is complete with respect to the POs, but $S_{nt}4$ isn't as it fails to prove any instance of Anti.

Note that to get the completeness result we didn't need to add as axioms instances of $n \rightarrow H(Pn \rightarrow n)$, so by the completeness result all such instances must be theorems. Finding syntactical proofs is not easy; the following one works by reducing the problem to a point where the Stopover Schema can be applied.

1. i	Assumption	
2. $\neg H(Pi \rightarrow i)$	Assumption	1
3. $P(Pi \wedge \neg i)$	2; TL	1, 2
4. $i \rightarrow HFi$	TL4	1, 2
5. HFi	1, 4; MP	1, 2
6. $HFi \wedge P(Pi \wedge \neg i)$	3, 5; PC	1, 2
7. $P(Fi \wedge (Pi \wedge \neg i))$	6; Inst	1, 2
8. $P(Pi \wedge (\neg i \wedge Fi))$	7; PC	1, 2
9. $i \wedge P(Pi \wedge (\neg i \wedge Fi)) \rightarrow F((\neg i \wedge Fi) \wedge Fi)$	Stopover Schema	1, 2
10. $F((\neg i \wedge Fi) \wedge Fi)$	1, 8, 9; MP	1, 2
11. $F(Fi \wedge \neg i)$	10; PC	1, 2
12. $G(Fi \rightarrow i)$	1; Anti Instance, MP	1, 2
13. $F((Fi \rightarrow i) \wedge (Fi \wedge \neg i))$	11, 12; Inst	1, 2
14. $F(i \wedge \neg i)$	13; PC	1, 2
15. \perp	14; TL	1, 2
16. $i \rightarrow H(Pi \rightarrow i)$	15, 2, 1 Discharge twice	

The modal logic of POs is axiomatised by $S_{nm}4Anti$, consisting of all instances of

$$\phi \rightarrow \Diamond \phi \quad (T)$$

$$\Diamond \Diamond \phi \rightarrow \Diamond \phi \quad (4)$$

$$n \rightarrow \Box (\Diamond n \rightarrow n) \quad (Anti)$$

added as axioms to K_{nm} . Again a light bulldozing argument gives the strong completeness result, and as usual for modal languages we can simplify the bulldozing process by choosing merely right unbounded order types for α .

5.4 Logics of linear frames

In this section we examine the logics of linear frames — that is, the logics of STOs and TOs. These logics are particularly natural ones: most importantly, we don't need to use a *double* generation process when proving completeness — the first generation step gives every nominal a home. Moreover, these frames drive an interesting wedge between modal and tensed languages. As we shall see, tensed languages are better than their modal counterparts at controlling nominals on linear structures. Finally, finite axiomatisations of these frames are possible: NOM_W or $SWEEP_W$ style schemas are no longer needed.

We begin by considering the cumulative effect of the following four schemas on Henkin frames:

$$\begin{aligned}
 FF\phi &\rightarrow F\phi & (4) \\
 F\phi \wedge F\psi &\rightarrow (F(\phi \wedge F\psi) \vee F(\phi \wedge \psi) \vee F(\psi \wedge F\phi)) & (RLin) \\
 P\phi \wedge P\psi &\rightarrow (P(\phi \wedge P\psi) \vee P(\phi \wedge \psi) \vee P(\psi \wedge P\phi)) & (LLin) \\
 n \vee Fn \vee Pn & & (Tr)
 \end{aligned}$$

4 imposes transitivity. Each of the Lin schemas defines local linearity in the appropriate direction, and moreover their inclusion as axioms imposes local linearity on Henkin frames.⁸ Now, it is clear that any point generated subframe of a transitive and locally linear frame is trichotomous, but this means that any axiomatisation which has *both* 4 and the Lin schemas is bound to have a Henkin frame whose point generated subframes are trichotomous. Hence if we include all instances of these three schemas as axioms, the first stage of our typical completeness proof generations will almost yield the straight lines we are looking for. In general, of course, we'll get 'quasi lines' instead of lines, as the structures we generate may contain clusters; but we know how to get rid of these

⁸A simple argument shows that given three Henkin frame points h_1, h_2 and h_3 such that $h_1 <_h h_2$ and $h_1 <_h h_3$, RLin guarantees that either $h_1 <_h h_2 <_h h_3$ or $h_1 <_h h_3 <_h h_2$. LLin works analogously leftwards. Proofs can be found in [16, page 103], and [63, pages 76–77].

and this choice of axioms clearly takes us some way towards the completeness theorems we want.

Double generation, however, would ruin everything. If we're forced to generate a second subframe to get the 'nominal bookkeeping' right, we lose trichotomy and the theorems we're looking for. This is where Tr comes in. Including its instances as axioms obviates the need to double generate. To see this first note that the troublesome sets of sentences $\{\neg Ei : E \text{ is an existential tense}\}$ are no longer consistent — they bluntly negate the new axioms, and we can no longer 'drive out' nominals. More directly, because one of n or Pn or Fn is in every MCS, *every nominal occurs in exactly one MCS in each point generated subframe of the Henkin frame*. Tr 'drives in' nominals. Note that this happens with *any* axiomatisation containing Tr, even K_nTr ; it does not depend on anything said earlier about 4 or the Lin schemas. Also note that although any of its instances defines trichotomy, the Tr schema does not impose trichotomy on the Henkin frame. The problem is the usual one: points of the Henkin frame not containing any nominal cannot be guaranteed to be 'trichotomously placed'. We will need the Lin schemas.

In short, the combination of 4, LLin, RLin and Tr gives us the basic Henkin frames we need to handle linearity: the first three schemas take care of the relational structure, while the last regulates the nominals and renders double generation unnecessary. Thus the completeness results we want are at hand. We begin with the STOs. Let LIN_n be the axiomatisation obtained by adding all instances of LLin, RLin and Tr to $I4$.

Theorem 5.4.1 *LIN_n is strongly complete with respect to the class of STOs.*

Proof:

Given a consistent set of sentences Σ , singly generate a verifying model. This is a model because, as all instances of Tr are axioms, the natural mapping on this frame is already an NTL valuation. Further, because of the instances of 4, LLin and RLin, the frame so generated will be transitive and trichotomous. The only thing that could go

wrong is that it's not irreflexive, but we can fix this by heavy bulldozing. We thus have verified our original set of sentences Σ on a STO and are through. \square

Another useful result can be obtained by adding all instances of D to LIN , giving the axiomatisation LIN, D . As D forces unboundedness on the Henkin frame, and as unboundedness is not affected by bulldozing the following result is clear:

Theorem 5.4.2 *LIN, D is strongly complete with respect to the class of all unbounded STOs.* \square

We can also axiomatise the reflexive analogs of the STOs, the TOs. Let LIN be the axiomatisation obtained by adding all instances of Llin, RLin and Tr to PO .

Theorem 5.4.3 *LIN is strongly complete with respect to the TOs.*

Proof:

Singly generate a model. This will be a trichotomous preorder by previous reasoning, and any proper clusters can be eliminated by light bulldozing. \square

Let's now consider the nominal modal logics of linear frames. As we shall see, these are a little peculiar. As a way into the problem, let's temporarily remain with tensed languages and consider what would happen if we dropped Tr from the three axiomatisations above. Clearly, because we're no longer driving nominals in to every point generated subframe, a double generation argument would be required to give each nominal a home. Thus these weakened axiomatisations give us 'local' versions of the previous completeness theorems. That is, they axiomatise the classes of all *locally* STOed frames, unbounded *locally* STOed frames, and *locally* TOed frames, respectively. Moreover, it is clear by soundness that these weakened axiomatisations really are weaker — none of them has Tr as a theorem. Now consider what happens with modal languages.

Let $I4.3$ be the nominal modal axiomatisation obtained by adding all (modal) instances of the schemas I, 4, and RLin to K_{nm} ; $I4.3D$ be $I4.3$ augmented by all instances

of $\Diamond\top$ (modal D); and $S4.3$ be the axiomatisation obtained by adjoining all (modal) instances of of Anti, 4 and T to K_{nm} . These axiomatisations are clearly the modal analogs of the *weakened* tense axiomatisations just considered: so far they contain no modal analog of Tr. Double generation arguments show that these axiomatisations capture the nominal modal logics of the obvious classes of frames. For example, $I4.3$ is strongly complete with respect to the class of all transitive, irreflexive and right locally linear frames.

The fun starts when we try to add a modal analog of Tr to capture the logics of STOs, unbounded STOs and TOs respectively. There simply doesn't appear to be anything more to say about these more restricted classes. Because we can't 'look back' in modal languages we clearly run the risk of 'leaving our nominals behind', or 'losing track of them'. In short, restricting ourselves to these classes doesn't seem to introduce any new validities and it seems reasonable to suppose that we can do in the modal case what couldn't be done in the tensed case: obtain a sharpening result by showing that $I4.3$, $I4.3D$ and $S4.3$, respectively, already suffice.

This can be shown and the proof is simple. In each case singly generate a weak model. As we have no 'driving in' schema we need to find a home for any unassigned nominals, but we *don't* do this by means of a second generation. Instead we adjoin a new point t^- that precedes all the points in our generated frame but is not itself preceded by any of them. (We stipulate that $t^- < t^-$ or $t^- \not< t^-$ depending on whether we want a reflexive or irreflexive model.) All unassigned nominals are assigned $\{t^-\}$. Clearly this model is transitive and trichotomous and verifies our original set of sentences. Bulldozing then gives us the precise sort of model required.

Generalising slightly yields the following. Let $\mathcal{M}(TranTr)$ be the class of all transitive and trichotomous models, and $\mathcal{M}(TranRLL)$ be the class of all transitive and right locally linear models. Then for any NML wff ϕ we have that $\mathcal{M}(TranTr) \models \phi$ iff $\mathcal{M}(TranRLL) \models \phi$. The right to left direction is immediate, and the reverse direction is similar to the proof above. That is, given some $\mathbf{M} \in \mathcal{M}(TranRLL)$ that falsifies ϕ at t , make a second model \mathbf{M}' by taking the subframe of \mathbf{M} modally generated by t and adjoin a new point t^-

that precedes all these points. Construct the new valuation in the obvious way, assigning t^- to any nominals which don't denote points in $gen(t)$ in the old valuation. Clearly M' also falsifies ϕ at t , and by construction $M' \in \mathcal{M}(TranTr)$. Note that any relational condition enjoyed by subclasses of $\mathcal{M}(TranRLL)$ that survives this 'modally generate and add a point at negative infinity' process yields completeness results of the sort noted above: irreflexivity, unboundedness and reflexivity are merely three examples. Modal languages won't be discussed further in this thesis.

Finite axiomatisations of at least the more obvious nominal tense logics of linear frames are possible. Define $\Diamond_t \phi$ to be $P\phi \vee \phi \vee F\phi$, and $\Box_t \phi$ to be $H\phi \wedge \phi \wedge G\phi$. On linear frames \Diamond_t is a 'somewhere' operator, and \Box_t an 'everywhere' operator — they are S5 possibility and necessity operators. Note that any instance of the Tr schema can be rendered as $\Diamond_t i$ — Tr says that any nominal must occur *somewhere*, which is precisely what modal languages couldn't say. Now define two new schemas:

$$\begin{aligned} n \wedge \Diamond_t(n \wedge \phi) &\rightarrow \phi && (\text{NOM}_{\Diamond_t}) \\ n \wedge \phi &\rightarrow \Box_t(n \rightarrow \phi) && (\text{SWEEP}_{\Box_t}) \end{aligned}$$

It is easy to see that the inclusion of either schema in place of NOM_w suffices to force unique occurrence on linear Henkin frames: we don't need all the path equations on linear frames because easy routes between points always exist. In turn, this means that by using axioms and a rule of substitution a finite axiomatisation can be given. The required rule is that of NTL *substitution*. We say that a wff ψ is obtained from a wff ϕ by NTL substitution iff ψ is the result of substituting arbitrary NTL wffs θ for variables in ϕ , and uniformly substituting nominals for nominals in ϕ .

The way these defined S5 operators successfully replace the usual existential and universal tenses suggests that it might be a good idea to introduce into our languages primitive S5 'everywhere' and 'somewhere' operators defined on arbitrary frames. This will be done in the following chapter.

5.5 Decidability and the finite model property

The logics we have considered so far are decidable. This is not particularly surprising: what is interesting is that in spite of an apparent obstacle, filtration methods can be used to prove this. The required adaptation brings to the fore a theme that will become increasingly prominent in the remainder of this thesis: the importance of classes of *models*, rather than classes of *frames*, when working with sorted intensional languages.

In standard tense languages filtrations provide a reasonably general method for establishing decidability. A typical proof runs as follows. Given an axiomatisation K_tS which we know to be (say, strongly) complete with respect to some class of frames \mathcal{T} , we attempt to show that it is also (weakly) characterised by the class of all finite frames in \mathcal{T} . In many important cases this can easily be proved using filtrations — as was mentioned in Chapter 2, unboundedness, trichotomy and reflexivity are inherited by all filtrations; taking Priorean filtrations ensures transitivity; and the whole point of working with filtrations is that filtrating through a finite set of sentences yields a model based on a finite frame — and when it can be shown we say that K_tS has the *finite frame property* with respect to \mathcal{T} . Given that the theorems of K_tS form a r.e. set — and in most cases of interest they will — this establishes decidability. For, if the finite frames in \mathcal{T} form an r.e. set — and again, in almost all cases of interest they will — searching through all the finite models based on these frames is an effective (if inefficient) procedure for generating all the *non*-theorems of K_tS , and thus the theorems of K_tS form a recursive set.

The apparent impediment to the application of filtration methods for this purpose in NTL is illustrated by the axiomatisation $I4D$. Any class of frames on which all its axioms are valid must consist solely of unbounded SPOs, hence no finite frame can validate its axioms and $I4D$ does not have the finite frame property. Thus the way appears blocked — but there is an interesting loophole. Although $I4D$ does not have the finite frame property it does have the *finite model property*. That is, it is possible to define a class of *models* \mathcal{M} such that $\vdash_{I4D} \phi$ iff $\mathbf{M} \models \phi$, for all $\mathbf{M} \in \mathcal{M}$. The class of models needed will shortly be described, for now merely note that the loophole we are exploiting does

not exist in standard languages: a well known theorem of Segerberg's states that if L is any classical modal logic, then L has the finite model property iff L has the finite frame property [98, page 33]. Thus Segerberg's theorem does not hold in NTL as $I4D$ is a counterexample.

The following arguments take this general form: certain classes of models are defined, and (weak) completeness results are proved for the axiomatisations we have already considered with respect to these new classes.⁹ What are these classes of models? Simply the most obvious class of models to which the Henkin models produced by our double generation process belong! That is, we abstract the required classes of models from the information we have about the global structure of our Henkin models, together with the local information we have as encapsulated in the Irreflexivity or the Simple Cluster Lemmas. Thus the weak completeness theorems for these classes of models have in effect already been proved by our double generation arguments, and the important step is to transfer these results to the finite case by means of filtration. This proves straightforward. Because of the 'additive inheritance' by filtrations of the more obvious relational properties, the real work takes place in our two base logics $I4$ and PO . We begin with $I4$.

Call $\mathbf{T} = \langle T, < \rangle$ an *irreflexivity containing* frame iff there is a $t \in T$ such that $t \not< t$. Call a valuation V on such a frame \mathbf{T} *irreflexivity respecting* iff $t \in V(i)$ is irreflexive for all nominals i . That is, irreflexivity respecting valuations are valuations on frames containing irreflexive points that send all nominals to irreflexive points. We call $\mathbf{M} = \langle \mathbf{T}, V \rangle$ an I_1 *model* iff \mathbf{T} is an irreflexivity containing frame and V an irreflexivity respecting valuation on \mathbf{T} . The class of all I_1 models is called $\mathcal{M}(I_1)$.

⁹We could prove strong completeness results with respect to these classes of models — indeed, as will soon become apparent, in effect we have already done so — but won't bother here. The real interest lies not with these rather trivial results, but in their sharpenings to the finite case by means of filtration, and in these sharpenings only weak characterisations are possible.

Lemma 5.5.1 *I4 is sound and complete with respect to the class of all transitive I_1 models. That is, $\vdash_{I4} \phi$ iff $\mathbf{M} \models \phi$, for all $\mathbf{M} \in \mathcal{M}(I_1) \cap \mathcal{M}(Tran)$.*

Proof:

(Soundness). The only axiom schemas that require checking are 4 and I, the others being universally valid. As the models \mathbf{M} we are considering are transitive, any instance of 4 is true in all such \mathbf{M} ; and as all nominals denote irreflexive points in \mathbf{M} , all instances of I are true in these \mathbf{M} . All three rules of inference preserve truth in a model, and so I4 is sound with respect to $\mathcal{M}(I_1) \cap \mathcal{M}(Tran)$.

(Completeness). This is shown by the first part of the completeness proof for I4 given above — the stage preceding bulldozing. Given any I4-consistent sentence ϕ , the double generation process yields a transitive Henkin model verifying ϕ at some point; moreover the Irreflexivity Lemma shows that every point containing a nominal is irreflexive, and thus this Henkin model is in $\mathcal{M}(I_1) \cap \mathcal{M}(Tran)$. \square

The important thing about this result is that it can be sharpened in the following fashion: we *don't* need all the models in $\mathcal{M}(I_1) \cap \mathcal{M}(Tran)$ to establish completeness — simply the finite ones will do.

Theorem 5.5.1 *I4 has the finite model property with respect to $\mathcal{M}(I_1) \cap \mathcal{M}(Tran)$. That is:*

$$\vdash_{I4} \phi \quad \text{iff} \quad \mathbf{M} \models \phi$$

for all finite $\mathbf{M} \in \mathcal{M}(I_1) \cap \mathcal{M}(Tran)$.

Proof:

Soundness follows from the previous lemma. The following filtration argument establishes completeness. By the previous lemma we know that given an I4-consistent sentence ϕ we can find an $\mathbf{M} \in \mathcal{M}(I_1) \cap \mathcal{M}(Tran)$ such that $\mathbf{M} \models \phi[t]$, at some point t . Now, if ϕ contains occurrences of nominals, define Σ^- to be

$$\{\phi\} \cup \{i \rightarrow \neg Fi : i \text{ occurs in } \phi\};$$

while if ϕ contains no occurrences of nominals choose any nominal — say i — and define Σ^- to be $\{\phi\} \cup \{i \rightarrow \neg Fi\}$. Let Σ be the smallest set of wffs containing Σ^- that is closed under subformulas. Form any Priorean filtration \mathbf{M}^f of \mathbf{M} through Σ such that for all nominals $j \notin \Sigma$, $V_f(j) = V_f(i)$, for some nominal $i \in \Sigma$. (This last point is just ‘bookkeeping’: nominals j such that $j \notin \Sigma$ aren’t important. What *is* important is that there is always at least one nominal in Σ . As we shall shortly see, this guarantees that there will be irreflexive points in the filtration.)

By the Filtration Theorem $\mathbf{M}^f \models \phi[E(t)]$. But \mathbf{M}^f is a model in the required class: clearly it is finite, because Σ is a finite set of sentences; and it is transitive because we took a Priorean filtration. Moreover \mathbf{M}^f *does* contain irreflexive points, and all nominals are assigned irreflexive points in this filtration. To see this, note that it follows from the definition of Priorean filtrations that:

$$\exists \phi (F\phi \in \Sigma \ \& \ \mathbf{M} \models \phi[t] \ \& \ \mathbf{M} \not\models F\phi[t]) \text{ implies } E(t) \not\prec_f E(t).$$

But for all nominals $i \in \Sigma$ — and there is always at least one — $Fi \in \Sigma$. Further, as our original model was in $\mathcal{M}(I_1)$, $\mathbf{M} \models i[t]$ means $\mathbf{M} \not\models Fi[t]$, and thus for all such points t , $E(t) \not\prec E(t)$. This means that all points in the filtration \mathbf{M}^f denoted by nominals are irreflexive, and we have our result. \square

Corollary 5.5.1 *I_4 is decidable.*

Proof:

Suppose $\not\models_{I_4} \phi$. By the previous theorem we can falsify ϕ on a *finite* transitive I_1 model. This means that routine search through all such finite models will eventually falsify ϕ . It is clear that a program to generate all the required *frames* can be written, and although we cannot generate all the models, for any formula ϕ we only need to consider the possible assignments that can be made to the finite number of atoms actually occurring in ϕ . Thus the set of wffs that are *not* provable in I_4 is an *re* set. But the set of wffs that *are* provable in I_4 is also *re* — we just keep systematically pumping out theorem from our axioms. (Alternatively, we could use the fact that there is an upper

bound, for any wff ϕ , on the size of the models that need to be searched. Failure to falsify ϕ on a model below this size means that ϕ must be a theorem.) Therefore the set of wffs provable in $I4$ is recursive. \square

Decidability for other extensions of $I4$ follow from this basic result.¹⁰ For example, $I4D$ is decidable because our double generation argument establishes that $\vdash_{I4} \phi$ iff $\mathbf{M} \models \phi$, for all unbounded $\mathbf{M} \in \mathcal{M}(I_1) \cap \mathcal{M}(Tran)$. As filtrations inherit unboundedness, the filtration described above for $I4$ establishes the finite model property for $I4D$ relative to this class of models, and decidability follows. Similarly, a result for LIN_s is obtained by noting that the double generation process establishes its completeness with respect to the class of all trichotomous $\mathbf{M} \in \mathcal{M}(I_1) \cap \mathcal{M}(Tran)$, and as filtrations inherit trichotomy we again have the finite model property and decidability.

Theorem 5.5.2 *$I4D$, LIN_s , and LIN_sD are all decidable.* \square

Thus we have a tool that works for at least some of the logics of interest above $I4$. I'll now sketch how to prove analogous results for logics above PO . The concepts we need are essentially those given above but with talk of 'simple clusters' replacing talk of 'irreflexive points'. That is, we define notions of *simple cluster containing frames*, *simple cluster respecting valuations*, and SC_1 models analogous to those given above, and denote the class of all simple cluster respecting models by $\mathcal{M}(SC_1)$. Our double generation method establishes that $\vdash_{PO} \phi$ iff $\mathbf{M} \models \phi$, for all $\mathbf{M} \in \mathcal{M}(SC_1) \cap \mathcal{M}(Pre)$, where $\mathcal{M}(Pre)$ is the class of all preordered models. Now for the key step:

Theorem 5.5.3 *PO has the finite model property with respect to $\mathcal{M}(SC_1) \cap \mathcal{M}(Pre)$. That is:*

$$\vdash_{PO} \phi \quad \text{iff} \quad \mathbf{M} \models \phi$$

for all finite $\mathcal{M}(SC_1) \cap \mathcal{M}(Pre)$.

¹⁰Indeed the weaker logic I must also be decidable. It has the finite model property with respect to $\mathcal{M}(I_1)$.

Proof:

Soundness is immediate, completeness again follows by filtration. Given a *PO*-consistent wff ϕ containing occurrences of nominals, define Σ^- to be

$$\{\phi\} \cup \{i \rightarrow G(Fi \rightarrow i) : i \text{ occurs in } \phi\},$$

and if ϕ is purely Priorean define Σ^- to be $\{\phi\} \cup \{i \rightarrow G(Fi \rightarrow i)\}$, for some selected nominal i . Let Σ be the smallest set of wffs containing Σ^- that is closed under subformulas.

Given a model M verifying ϕ , filtrate M through Σ to form M^f . By the Filtration Theorem this model also verifies ϕ . Now M^f is finite and transitive, and as filtrations inherit reflexivity automatically we have that M is a finite preorder. If it contains simple clusters, and if all nominals denote these, we are through.

Now we know from Chapter 2 that any nominal in Σ is assigned a point $E(t)$ in the filtration which is a set containing a single point t of the original model. Suppose that there is another point $E(t')$ in the filtration such that $E(t) <_f E(t')$ and $E(t') <_f E(t)$. As $M \models G(Fi \rightarrow i)[t]$ (which it does, because in the original model both i and $i \rightarrow G(Fi \rightarrow i)$ are true at t), and as $G(Fi \rightarrow i) \in \Sigma$, $M^f \models G(Fi \rightarrow i)[E(t)]$ by the Filtration Theorem. But as $E(t) <_f E(t')$ and $G(Fi \rightarrow i) \in \Sigma$, this means that $M^f \models Fi \rightarrow i[E(t)]$. But $E(t') <_f E(t)$, and as $M^f \models i[E(t)]$ and $Fi \in \Sigma$, we have that $M^f \models Fi[E(t')]$, which by modus ponens yields $M \models i[E(t')]$. But this means that $E(t') = E(t)$, and thus all points in the filtration denoted by nominals are simple clusters. \square

Corollary 5.5.2 *PO and LIN are decidable.* \square

To summarise, the Henkin models we build by our double generation process embody important information which can be read in (at least) two ways: in terms of frames or in terms of models. Under the first reading our task is to realise the information in Henkin frames in relational terms, and we search for ‘structural rectification’ techniques that lead to frame characterisation theorems. Under the second reading our task is to condense

the information into a smaller model, and we search for size rectification techniques, and finite model results. Both tasks exist in standard intensional languages of course, but in such languages purely structural matters tend to dominate — Segerberg’s Theorem is an example of this. In sorted frameworks, on the other hand, constraints on the frame order $<$ are simply one of two important parameters of variation; the other is the constraints in force on our valuations. This raises a theme that will be explored in the remainder of this thesis: the need to abandon the frame centred view of standard languages and pay more attention to models. For now I’ll merely say that the results of this section show that we need read flexibly the information in Henkin frames. This noted, let us resume our examination of linear time logics.

5.6 Special structures

In this section we axiomatise the frames **Q**, **R**, **Z** and **N**. The four axiomatisations are pleasantly uniform: each arises from the corresponding standard tense logical axiomatisation by adding all instances of the schemas NOM_W , I and Tr . Either of the new finite minimal schema for linear frames could be used in place of NOM_W .

As far as *proving* these results is concerned, the simple nature of the extensions required indicates that we shouldn’t have to alter the standard proofs much. In fact, to secure the completeness results all we really have to do is ensure that the various rectification techniques applied in the course of these proofs preserve unique occurrence. They do: the standard proofs use various permutations of heavy bulldozing, Priorean filtration and some (simple) new devices. As usual the Irreflexivity Lemma tells us that bulldozing will not be troublesome, constructing filtrations as was shown in the proof that $I4$ has the finite model property ensures that these work appropriately, and the new devices turn out to be unproblematic. In short, most of the work has already been done.

The NTL logics of these frames are also decidable, though only in the case of **Q** will we be able to prove this by means of filtration arguments. The problem with **Z**, **N** and **R** is that although filtrations are used in their completeness proofs, and although in all three

cases these give rise to natural finite classes of $\mathcal{M}(I_1)$ models — ‘dumbbells’, ‘lollypops’ and ‘necklaces’ respectively — the respective logics are not sound with respect to these classes. Nonetheless, decidability results for the other three frames follow from more general considerations. Briefly, we know that NTL validity corresponds to the validity of certain Π_1^1 L_2 wffs. Rabin-Gabbay arguments show that this fragment of \mathbf{Z} and \mathbf{N} ’s higher order theory is decidable, and more recent results by Burgess and Gurevich show that the same holds for \mathbf{R} . Further details of both proofs may be found in [17].

In the following I sketch the strong completeness and decidability results for \mathbf{Q} and then indicate, following Robert Goldblatt, how a (weak) completeness result may be obtained for \mathbf{R} . Then weak completeness results for \mathbf{Z} and \mathbf{N} are presented, with the emphasis on the result for \mathbf{Z} . As with so much else in this chapter, the account given essentially follows Segerberg’s [97]. However at a number of indicated places simplifications introduced by Goldblatt [40, Chapter 8] have been incorporated.

LIN,DA is the axiomatisation obtained by adding to LIN,D all instances of

$$F\phi \rightarrow FF\phi \quad (\mathbf{A}),$$

the schema corresponding to density. Note that when we write out in full the schemas LIN,DA contains in addition to those of K_t , we find we have 4, LLin, RLin, D and A — the schemas that axiomatise the tense logic of \mathbf{Q} in standard languages — and in addition NOM_W , I and Tr. As we shall now sketch, LIN,DA does axiomatise the tense logic of \mathbf{Q} in languages of NTL.

Theorem 5.6.1 *Let the class of all dense, unbounded, transitive and trichotomous models be called $\mathcal{M}(Q_W)$. Then:*

1. LIN,DA is sound and strongly complete with respect to $\mathcal{M}(I_1) \cap \mathcal{M}(Q_W)$;
2. LIN,DA is strongly complete with respect to the class of all unbounded dense STOfed frames;
3. LIN,DA is strongly complete with respect to the frame \mathbf{Q} .

4. LIN,DA has the finite model property with respect to $\mathcal{M}(I_1) \cap \mathcal{M}(Q_W)$; and
5. LIN,DA is decidable

Proof:

(1). Soundness is clear. Next, given a LIN,DA -consistent set of sentences Σ , singly generate a verifying model \mathbf{M}_1 . The occurrences of 4, LLin, RLin and D ensure its transitivity, trichotomy and unboundedness respectively; and it is simple matter to show that A guarantees its density — a proof may be found in [16, page 105]. Moreover each nominal occurs somewhere, and that somewhere must be irreflexive by the Irreflexivity Lemma, hence $\mathbf{M}_1 \in \mathcal{M}(I_1)$. All the conditions are met, and the existence of \mathbf{M}_1 yields strong completeness.

(2). Heavily bulldoze \mathbf{M}_1 , taking care to first embed each cluster C_δ of \mathbf{M}_1 in some set C_δ^\dagger that is unbounded and dense — δ indexed copies of \mathbf{R} will do. This produces a model \mathbf{M}_2 that also verifies Σ . By the reasoning of previous proofs \mathbf{M}_2 is an unbounded STO, and in addition it is dense. (This boils down to the observation that when bulldozing we took care to replace clusters by suitable dense sets.) Thus the existence of \mathbf{M}_2 establishes the second part.

(3). Although ‘Q-like’, \mathbf{M}_2 in general will be too large. However by our correspondence results \mathbf{M}_2 is also a model for the set of \mathbf{L}_1 wffs obtained by standardly translating all the wffs in Σ . Using the Downward Löwenheim Skolem Theorem for \mathbf{L}_1 establishes the existence of a *countable* dense unbounded model for these wffs, and this countable model is also a model for Σ . However there can be only one such frame (up to isomorphism) on which this model is based — namely \mathbf{Q} — and we are through.

(4). Given a LIN,DA consistent sentence ϕ make a model \mathbf{M} for it by the process described in part 1. We know that $\mathbf{M} \in \mathcal{M}(I_1) \cap \mathcal{M}(Q_W)$. Form Σ^- and Σ by the process described in the proof that $I4$ has the finite model property, and Prior filtrate \mathbf{M} through Σ to form the finite model \mathbf{M}^f . This model verifies ϕ and moreover has all the desired relational properties save possibly density. In fact it must be dense as well: Segerberg’s

Lemma 1.1 [97, page 307] shows that \mathbf{M} cannot contain adjacent irreflexive points, and thus the existence of \mathbf{M}' establishes part 4.

(5). Part 4 yields an re set of finite models on which any non-theorem can be falsified.

□

The use of the Löwenheim Skolem Theorem to prove part 3 can be avoided. A method of building the required model in stages is given by de Jongh et al in [50], and has been adapted to D-logic by Maarten de Rijke in [89, pages 36–38]. The method is essentially a sophisticated version of the Makinson technique introduced in the previous chapter. It is very elegant, and the only reason it has not been adopted here is because of the length of such a proof.

$Th(\mathbf{R})$ in standard languages can be axiomatised by adding all instances of the schema

$$\Box_i(H\phi \rightarrow FH\phi) \rightarrow (H\phi \rightarrow G\phi) \quad (\text{Cont})$$

to the standard axiomatisation of \mathbf{Q} , as Goldblatt proves in [40, pages 66–69].¹¹ An examination of Goldblatt's proof shows that $LIN, DACT$ axiomatises the $Th(\mathbf{R})$ in languages of NTL. Briefly, given a $LIN, DACT$ -consistent sentence ϕ generate a model for it and filtrate this model to form \mathbf{M}' as described in part 4 above. \mathbf{M}' may not do as it stands. We want to be able to 'expand' it into a copy of \mathbf{R} — that is, we want to define a p-morphism from an \mathbf{R} based model to \mathbf{M}' . This is impossible if \mathbf{M}' contains adjacent clusters. However it is possible to interpolate new irreflexive points between any such cluster in a truth preserving way; this is the tricky part of the proof, and here use must be made of Cont. For our purposes all we need to observe is that in Goldblatt's construction none of these interpolated points are assigned nominals, thus the new 'gap filled' finite model has the unique occurrence property and (by Goldblatt's p-morphism)

¹¹Segerberg seems to have been the first to show that $Th(\mathbf{R})$ could be axiomatised by adding schemas to the axiomatisation of \mathbf{Q} . For this purpose Segerberg added $(F\phi \leftrightarrow H\phi) \rightarrow (\Diamond_i\phi \rightarrow \Box_i\phi)$, together with its mirror image. The proof that this suffices may be found in [97, pages 315–316].

is equivalent to M^f . This new model can be successfully expanded to an R based one and weak completeness follows.

Theorem 5.6.2 *$LIN, DACont$ is weakly complete with respect to R* □

Let us now axiomatise the NTL logics of the discrete structures Z and N . Because of the failure of compactness noted in the previous chapter we cannot hope to prove strong completeness theorems for these frames, but weak completeness results are forthcoming. Let LIN, DZ be the axiomatisation obtained by adding all instances of the schemas

$$\begin{aligned} G(G\phi \rightarrow \phi) &\rightarrow (FG\phi \rightarrow G\phi) & (Z_r) \\ H(H\phi \rightarrow \phi) &\rightarrow (PH\phi \rightarrow H\phi) & (Z_l) \end{aligned}$$

to LIN, D . These modified Löb schemas are familiar from Chapter 3, where they were used together with other LIN, D axioms to define Z up to isomorphism. We now give a reasonably detailed proof that LIN, DZ successfully captures $Th(Z)$. The only important step missing is the proof of the Z-Lemma, and this may be found in [40, page 55].¹²

Theorem 5.6.3 *LIN, DZ is weakly complete with respect to Z .*

Proof:

Given a LIN, DZ -consistent sentence ϕ , singly generate a verifying Henkin model M_1 . By previous reasoning $M_1 \in \mathcal{M}(I_1)$ and is transitive, trichotomous and unbounded. Define Σ^- and Σ as in the proof that $I4$ has the finite model property, and Prior filtrate M through Σ to form M^f . As $M^f \in \mathcal{M}(I_1)$, none of the cluster rectification techniques applied below will effect the nominals.

M^f is a *finite* linear sequence consisting of some mixture of irreflexive points and clusters. Each cluster can only have finitely many members. Because M^f is unbounded

¹²The proof of this lemma is where the Z axioms are used, hence its name. Goldblatt's Z-Lemma is a simplification of Segerberg's Lemma 1.6 [97, pages 309–310].

and finite it must have a cluster at each end. Call the cluster at the beginning of time the *first* cluster, and the cluster at the end of time the *last* cluster. Note that these two clusters are distinct as our filtration method guarantees the existence of at least one irreflexive point in M^f , and this must lie between the first and last cluster. Call any cluster that is neither the first nor the last cluster an *intermediate* cluster. The completeness proof proceeds in two stages. First we must remove all the intermediate clusters to create a dumbbell shaped model M^d . We then replace the first and last cluster in M^d by a backward pointing and a forward pointing copy of N respectively, yielding a frame isomorphic to Z . Justifying both stages requires use of the Z-Lemma.

In our terminology this lemma says that if C is a non-first (non-last) cluster in M^f such that $E(t) \in C$, then if $G\phi \in \Sigma$ ($H\phi \in \Sigma$), and $M_1 \not\models G\phi[t]$ ($M_1 \not\models H\phi[t]$), then there exists a point t' in M_1 such that $M_1 \not\models \phi[t']$ and $c <_f E(t')$ for all $c \in C$ ($E(t') <_f c$ for all $c \in C$). Note that by the Filtration Theorem this means that $M^f \not\models \phi[E(t)]$.

This enables us to get rid of all intermediate clusters. Choose such a cluster C and impose an arbitrary strict total ordering on its points and then 'sew in' this STO into the frame underlying M^f in place of C . (Or, to express matters another way, weaken the equivalence relation on this cluster to some strict total ordering. That is, throw away certain pairs $\langle c, c' \rangle$ from $<_f \cap C^2$ including all reflexive loops $\langle c, c \rangle$ until a STO is obtained.) Call this new model M^{f-1} . We now seek to prove by induction that M^f is equivalent to M^{f-1} . This is simple save for one case: given a formula of the form $H\phi$ or $G\phi$ that is *false* at some point in M^f , how do we know it remains false in M^{f-1} ? (The corresponding point in M^{f-1} is related to fewer potential falsifiers due to the weakening.) The Z-Lemma provides the answer. For any formula of the form $G\phi$ or $H\phi$ in Σ it guarantees the existence of a point $E(t')$ *outside* the weakened cluster that falsifies ϕ . This, together with the fact that both models are transitive, removes the only impediment to the induction and the two models are equivalent.

Remove all the (finitely many) clusters in this fashion, and call the resulting dumbbell shaped model M^d . We now want to eliminate the first and last cluster in the manner previously described. Consider the last cluster. *Modally* bulldoze it by embedding this

finite cluster identically into itself and choosing N for the order α . This means that the right end of our new model looks like:

$$c_1, c_2, \dots, c_n, c_1, c_2, \dots, c_n, c_1, c_2, \dots, c_n, \dots$$

where the c_m ($1 \leq m \leq n$) are all and only the elements of the last cluster. The usual map f from the resulting model to M^d is a *modal* p-morphism, thus an inductive proof of the equivalence of the two models will not founder on wffs of the form $G\phi$. What about wffs of the form $H\phi$? Again we use the Z-Lemma: the last cluster was not first, so the lemma applies for formulas of the form $H\phi$, and thus the equivalence induction goes through. We then perform the ‘mirror image’ process on the first cluster of this model, and the result is a verifying model for ϕ based on a frame isomorphic to Z . \square

We now axiomatise $Th(N)$. Let LIN, D, Z, W_l be the axiomatisation obtained by dropping D_l and Z_l from LIN, DZ and adding all instances of

$$H(H\phi \rightarrow \phi) \rightarrow H\phi \quad (W_l),$$

the leftward Löb schema.

Theorem 5.6.4 LIN, D, Z, W_l is weakly complete with respect to N .

Proof:

Similar to the previous theorem. The major change is that the inclusion of W_l allows us to prove a ‘W-Lemma’ which is stronger than the Z-Lemma as it allows us to weaken *any* cluster in the manner described above without affecting formulas of the form $H\phi$. This means that we can eliminate the first cluster in M^f , as well as all the intermediate ones, yielding a lollypop shaped model. We then modally bulldoze the last cluster as described in the previous proof and have a model based on a frame isomorphic to N . \square

The nominal tense logics of the structures $\langle Q, \leq \rangle$, $\langle Z, \leq \rangle$, $\langle N, \leq \rangle$ and $\langle R, \leq \rangle$ can be axiomatised by adding all instances of the *Anti* and *Tr* schemas to the respective standard tense logical axiomatisations. The reader will be able to verify this by modifying either Segerberg’s or Goldblatt’s proofs.

5.7 The COV rule

Gargov, Passy and Tinchev have investigated using a rule they call COV in systems of nominal modal logic.¹³ This rule is useful because when assumed as part of an axiomatisation it often allows models with strong valuations to be constructed directly, thereby avoiding the cluster analysis methods of this chapter.

COV can be stated as an infinitary rule of inference. It then takes the following form: if for all nominals i we can show $L \neg i$, then we can deduce $L \perp$. (Here L is one of the necessity forms defined at the end of the last chapter.) As the authors mention, however, the rule is not intrinsically infinitary as the following version is equivalent: if we can deduce $L \neg i$ for any nominal i not occurring in L , we can deduce $L \perp$. The rule is sound in any model enjoying a strong valuation. Note that the use of COV does not result in new theorems in our axiomatisations (at least for languages with a non-finite number of nominals) by the equivalence between strong and ordinary validity proved in Chapter 2; rather, COV encapsulates something about reasoning in a strong model.

In [34] the minimal completeness theorem for nominal modal logic is proved with the aid of COV. The proof proceed in two stages. Firstly, following [39, page 73 – 76], a maximal theory is built, and following this certain points are equivalenced. This builds a model, but does rather more: because of the COV rule it is possible to show that the valuation is strong. In the later paper [35], the authors mention two extended completeness results which can be proved with its aid: they can axiomatise the nominal modal logics of the STOs and Q , and doubtless a wide variety of other logics too. However

¹³The rule was first introduced in [34] and used there to axiomatise the minimal nominal modal logic. Extended completeness results are not treated there, but are mentioned in the recent [35]. The newer paper also extends the use of the rule to incorporate the shifter operator discussed in the following chapter, and goes on to discuss hierarchies of COV rules for this system, but these developments cannot be considered here.

the use of the rule does not always guarantee that a strong model will be formed; the later paper gives a counterexample involving N .

I have not yet tried proving tense logical completeness results using a tense logical analog of COV. For the sake of discussion I'll assume that the natural analog will yield simpler completeness results for many of the classes of frames considered in this chapter. What import does this have for the completeness results above? Basically, viewed from the COV perspective, they amount to proofs that on certain classes of frames the use of this rule is eliminable. (At present, even in NML, there don't seem to be any general results stating when the use of the rule can be dispensed with.) There are good reasons for believing that COV will prove useful, nonetheless in general I think it is desirable to try and form as simple a deductive base as possible. Thus, even if easier completeness proofs become available using the new rule, the above results retain independent interest. Note that the decidability results proved above transfer to nominal modal logics using COV. For the systems given above, these results seem to be new. (In the later paper there is only a brief comment to the effect that if a purely nominal modal logic is decidable, its minimal extension — that is, the logic obtained by straightforwardly extending to the new language — must be also [35, page 21]. Of course this is true; but the more interesting results lie with the failure of Segerberg's Theorem and non-minimal extensions.)

The COV rule is a beautiful idea, and it may prove very valuable in more complex systems. At present I'm investigating using it in interval based logics. As we shall see in the final chapter, cluster analysis methods become complex in the richer intervallic setting, and the COV rule may be a useful addition to these investigations.

Chapter 6

Interval Nominals, the Shifter, and Sorting Generalised

In this chapter we turn from our examination of nominals to wider issues. First we introduce a new sort, interval nominals, which are name for intervals in the same fashion that nominals are names for points. Following this we add a new operator to our languages, the shifter, which allows us to exploit the referential abilities of nominals and interval nominals more directly. The resulting languages — tensed languages with nominals, interval nominals, and the shifter — are called fully referential tensed languages, or languages of TREF, and form the basis for the applications in natural language semantics investigated in the next chapter. In the final section we consider the idea of sorting in more generality, and examine some of the questions raised by such general systems.

6.1 Interval nominals

Interval nominals are a sort whose interpretation is constrained to be true at precisely the points in some unbroken stretch of time, or interval. Just as nominals can be thought of as names for instants, interval nominals can be thought of as names for intervals. The ability to refer to intervals will prove useful in the following chapter when we consider applications in natural language semantics. Briefly, natural language contains a variety

of mechanisms for referring to such stretches of time and insisting that some item of information is associated with it. For example, we can use indexical adverbials and say such things as ‘Yesterday Harold gave up fishing’, or a proper name as in ‘On Saturday John went fishing’. The inclusion of names for intervals in our object languages is a preliminary step towards an analysis of the above type of example.

But although in a clear sense we will now be ‘doing interval logic’, and are working towards ‘doing interval semantics’, the way in which we will do these things differs fundamentally from the norm that has become established over the past twelve or so years. During this time there has been considerable interest in the formal semantics community in constructing interval based semantics for natural languages.¹ Over the same period a number of logicians became interested in developing interval based tense logics.² But what could be meant by ‘interval tense logic’? What these authors have in common is that they understand by this phrase a radical change to the Kripke style intensional truth definition: tensed languages are no longer evaluated at points but rather at intervals of time. This move gives rise to all sorts of (interesting) problems of interpretation; see [16, pages 126–127] for a discussion of these issues. For example Humberstone [48] in one of the earliest papers in this tradition has suggested that this shift calls for a reinterpretation of the logical constants; he suggests an ‘intuitionistic’ negation. The seemingly innocuous change to intervallic evaluation thus has profound consequences, and Burgess’ comment that “the whole problem of interpretation for period-based tense logic deserves more careful thought” [16, page 127] seems a fair one. Indeed a number of authors, notably Antony Galton [28] [29], and Pavel Tichy [106], have heavily criticised the whole interval based approach.

¹Pioneering work in this tradition was done by David Dowty [25], and M.J. Cresswell [22]; for more recent work which combines these ideas with those of Discourse Representation Theory, see Partee’s [75].

²Early contributors include Humberstone [48] and Röper [92]. Some later results may be found in Burgess’ [15], and a full survey is provided in [5].

My own view — at least as far as logical issues are concerned — is that interval based languages are too interesting to ignore, and in the final chapter of this thesis some preliminary results concerning interval based systems with referential sorts are presented. However as far as applications of interval based languages are concerned, matters are not so clear to me. I certainly believe that ordinary interval based systems, which don't allow reference to intervals, are inadequate for such applications for precisely the same reason that standard (non-referential) Priorean languages are.

With this motivation, to work.³ By a language of interval nominal logic \mathcal{L} is meant the selection of two (countably infinite) disjoint sets of symbols, VAR (whose members are typically represented by p, q, r, \dots), and INOM (whose members are typically represented by e, d, c, \dots). The members of INOM are called *interval nominals*, and as usual the members of VAR are called variables. Wffs are formed in the normal fashion by boolean combination and application of tense operators. In short, as with NTL, the only syntactic change is that we have a second atomic sort. We take the usual syntactic definitions as read and turn to semantics. Unlike languages of NTL, these languages are only interpreted on frames that contain *intervals*; sets of points that 'look like an interval of time'. What do we mean by this? The decision that has been adopted here is that these are sets which both form an unbroken flow, and lie in a straight line. More precisely, we are interested in (non-empty) subsets which are both *convex* and *trichotomous*.

Let $\mathbf{T} = \langle T, < \rangle$ be any frame. By $Conv(\mathbf{T})$ is meant the set of all non-empty *convex subsets* of \mathbf{T} , that is:

$$Conv(\mathbf{T}) = \{S \subseteq T : \forall s_1 s_2 \in S \forall t \in T (s_1 < t < s_2 \Rightarrow t \in S) \text{ and } S \neq \emptyset\},$$

and by $Tr(\mathbf{T})$, the set of all non-empty *trichotomous subsets* of \mathbf{T} , that is:

$$Tr(\mathbf{T}) = \{S \subseteq T : \forall s_1 s_2 \in S (s_1 < s_2 \text{ or } s_1 = s_2 \text{ or } s_2 < s_1) \text{ and } S \neq \emptyset\}.$$

³In general in this chapter the presentation will be swifter than in previous work, and usually definitions will be less formal. The more precise definitions given for NTL in earlier chapters are intended as a guide for the work given here.

By $Int(\mathbf{T})$, the intervals of \mathbf{T} , is meant $Conv(\mathbf{T}) \cap Tr(\mathbf{T})$.

The first thing to note is that not all frames contain intervals. For example, let $\mathbf{B}_\omega = \langle B, < \rangle$ be the binary branching tree of depth ω . Note that \mathbf{B}_ω is intransitive, irreflexive, asymmetric and unique up to isomorphism. Let $SC(\mathbf{B}_\omega)$ be the symmetric closure of this frame. There are no intervals on $SC(\mathbf{B}_\omega)$: there is only one convex subset on $SC(\mathbf{B}_\omega)$ — namely B itself — and this is not trichotomous, hence $Int(SC(\mathbf{B}_\omega)) = \emptyset$. This example also shows that singleton subsets of a frame need not be intervals. Nonetheless there is a rich class of frames that do contain intervals. Note that any antisymmetric or asymmetric frame \mathbf{T} contains intervals, since any singleton subset $\{t\}$ satisfies both requirements. Moreover, on such classes as the SPOs and POs not only do intervals exist, but they have a number of pleasant global properties, as will be noted presently.

Let I be the class of frames \mathbf{T} such that $Int(\mathbf{T}) \neq \emptyset$. We interpret languages with interval nominals on frames in I , and the constraint we impose on functions V from the atoms of our language to $Pow(\mathbf{T})$ (where $\mathbf{T} \in I$) if they are to be valuations is that $V(e) \in Int(\mathbf{T})$, for all interval nominals e . We call such a function an *interval nominal valuation*, or simply an *interval valuation*. This atomic level constraint is the only change we make; all higher level definitions such as truth in a model and validity are as for languages of NTL. It is useful to have the notion of a *weak interval valuation*. For languages of nominal interval logic this means a function V that either is a valuation, or only fails to be one because it assigns \emptyset to some of the interval nominals. Note that weak interval valuations make sense on the class of all frames \mathcal{U} .

The model theory of interval nominals is rich and complex. We begin by giving some examples of newly definable classes of frames.

Firstly, e defines the class of frames \mathcal{T} such that for all $\mathbf{T} \in \mathcal{T}$, \mathbf{T} is trichotomous and $conv(\mathbf{T}) = \{T\}$. For suppose \mathbf{T} is such a frame. Then any valuation V must assign T to e , and thus e is true everywhere in any valuation. Conversely suppose that \mathbf{T} is not in this class. If \mathbf{T} is not trichotomous there are points $t, t' \in T$ such that $t \neq t'$, $t \not\prec t'$ and $t' \not\prec t$, and clearly in any valuation V we cannot have both t and t' are in $V(e)$. Thus in any valuation either $V(e, t) = -1$ or $V(e, t') = -1$. So suppose \mathbf{T} is

trichotomous and that there is an $S \subseteq T$ such that S is convex and $S \neq T$. Note that as T is trichotomous, S must be also, and thus S is an interval. Let V be any valuation such that $V(e) = S$. By assumption there exists a $t \in T \setminus S$, hence $V(e, t) = -1$ and we are through. This class of frames is not definable in a purely Priorean language: any element of this class is a connected frame, thus the class is not closed under the formation of disjoint unions. Conversely, the definability of this class using interval nominals shows that the disjoint union result fails for languages with interval nominals. In passing note that in languages with weak interval nominals $e_w, d_w, c_w \dots$; the wff $\neg e_w$ defines the class of all frames without intervals.

Fe defines the class of frames \mathbf{T} such that for all $S \in \text{Int}(\mathbf{T})$, $t \in T$ implies there exists an $s \in S$ such that $t < s$. That Fe is valid on all such frames is clear. Conversely, if \mathbf{T} is not in this class then there exists an $S \in \text{Int}(T)$ and $t \in T$ such that for all $s \in I$, $t \not< s$. Any valuation V which assigns S to e falsifies Fe at t . Again this class is not definable in purely Priorean languages because each frame in this class is connected.

The next example is rather more interesting. First some notation. Let $\text{Tri}(y, z)$ be the L_0 predicate $(y < z \vee y = z \vee z < y)$. Then on *transitive* frames containing intervals, $e \rightarrow F\neg e$ defines the condition $\forall x \exists yz (x < y \wedge x < z \wedge \neg \text{Tri}(y, z))$.

To see this first suppose that \mathbf{T} is a transitive frame satisfying the above condition. Let V be any valuation on \mathbf{T} , $t \in T$ and suppose $\langle \mathbf{T}, V \rangle \models e[t]$. By assumption there are t_1 and t_2 such that $t < t_1$ and $t < t_2$ and $\neg \text{Tri}(t_1, t_2)$. It is impossible that both t_1 and $t_2 \in V(e)$, for this would mean $V(e)$ was not trichotomous and that V was be a valuation; thus $V \models F\neg e[t]$.

For the reverse direction, suppose that \mathbf{T} is transitive and does not satisfy the given condition. That is, $\exists x \forall yz (x < y \wedge x < z \rightarrow \text{Tri}(y, z))$. Let $S_x = \{x\} \cup \{t \in T : x < t\}$. Trivially S_x is trichotomous. It also must be convex, for suppose there was an $w \in T$ such that for some $s, s' \in S_x$, $s < w < s'$ but $w \notin S_x$. Now if $s = x$, then we have $x < w$ and $w \notin S_x$, a contradiction. On the other hand, if $s \neq x$ then as $s \in V(e)$ we have $x < s$, and as \mathbf{T} is transitive and $s < w$ we have $x < w$; as $w \notin S_x$, this yields another contradiction. So there can be no such w and S_x is convex and hence an interval. Let

V be any valuation such that $V(e) = S_x$. Any such valuation falsifies $e \rightarrow F\neg e$ at x , establishing the result.

Purely Priorean languages cannot define this class on the transitive frames. Let \mathbf{B}_ω be the uniformly branching binary tree of depth ω , and let $TC(\mathbf{B}_\omega)$ be its transitive closure. Then the function $f : T(\mathbf{B}_\omega) \rightarrow \mathbb{N}$ that assigns every node its depth is a p-morphism. Hence no purely Priorean wff ϕ can define the above condition on transitive frames, for if such a ϕ existed it would have to be valid on $TC(\mathbf{B}_\omega)$, and f would then force ϕ to be valid on \mathbb{N} as well.

An example of an \mathbf{L}_0 condition not definable in a purely Priorean language, but definable with interval nominals, is $\forall t \exists t' (t < t' \wedge t' < t)$. This can be defined by $e \rightarrow Fe$; the example is van Benthem's. Firstly suppose that \mathbf{T} is a frame satisfying the condition, and suppose that $V(e, t) = 1$. By assumption there is a $t' \in T$ such that $t < t'$ and $t' < t$, and as $V(e)$ must be convex, $t' \in V(e)$. But this means that $V(Fe, t) = 1$. Conversely suppose that \mathbf{T} does not satisfy the condition; thus there is a $t \in T$ that is not symmetrically related to any point $t' \in T$. (In particular, $t \not< t$.) But this means that $\{t\}$ is an interval, and any valuation V which assigns $\{t\}$ to e refutes $e \rightarrow Fe$ at t .

These examples should convince the reader that interval nominals are model theoretically complex. How are they to be modeled in classical languages? The fundamental correspondence is this: interval nominal validity corresponds to the validity of certain \mathbf{L}_2 wffs. Let's first impose a little more order on \mathbf{L}_2 . Divide its predicate variables into two disjoint countably infinite sets. One set, whose elements are written P_1, P_2, P_3, \dots ; corresponds to the propositional variables in the usual fashion, while the other set, whose elements we write as E_1, E_2, E_3, \dots ; corresponds to the interval nominals. In order to allow the second set of predicate variables $\{E_m : m \in \mathbb{N}\}$ to capture the effect of interval nominals we first define for each predicate variable E in this set a second order predicate $I(E)$ by:

$$\forall t_1 t_2 (Et_1 \wedge Et_2 \rightarrow Tri(t_1, t_2)) \wedge \forall t_1 s t_2 (Et_1 \wedge Et_2 \wedge t_1 < s < t_2 \rightarrow Es)$$

We then extend the standard translation to languages with interval nominals by adding

the new atomic clause:

$$ST(e) = Et_0.$$

We then have:

$$\mathbf{T} \models \phi \text{ iff } \forall P_1 \dots \forall P_m \forall E_1 \dots \forall E_n \forall t_0 (I(E_1) \wedge \dots \wedge I(E_n) \rightarrow ST(\phi)),$$

and thus the validity of an arbitrary wff ϕ of a language with interval nominals corresponds to the validity of certain Π_1^1 wffs of \mathbf{L}_2 .

What preservation results do we have? Clearly validity is preserved under the formation of generated subframes. As with languages of NTL we need to be a little careful when formulating the notion of a generated *submodel* — once more we need the notion of a valuation being *in* a generated subframe — but this is straightforward.

As we have already noted, the disjoint union result fails, but we can say more than this. Recall that in Chapter 3 we gave a necessary and sufficient condition for an NTL formula to have its validity preserved under the formation of disjoint unions. We showed that the validity of an NTL formula ϕ is so preserved iff ϕ^\perp is valid in all component frames making up the disjoint union, where ϕ^\perp is the result of conjoining all possible substitutions of \perp for nominals in ϕ . Reflection shows that that this result has little to do with with nominals as ‘names for points of time’, but rather reflects two general facts about the constraint on NTL valuations. Firstly, nominals are required to be true *somewhere* in every valuation — we might say that nominals are an *existential sort*. Note that interval nominals are also existential sorts, and that neither weak nominals nor weak interval nominals are. Secondly, note that the constraint in force on both nominals trivially requires that the denotation of any nominal *must lie wholly within a single point generated subframe of any frame*. We might express this by saying that nominals are a *single island sort*. Note that interval nominals are also a single island sort, as are weak nominals and weak interval nominals; and that a sort constrained to denote precisely two distinct points would not be. It is clear that the necessary and sufficient condition given in Chapter 3 for nominal validity to be preserved under disjoint unions, really reflects the fact that nominals are an existential single island sort. The definition of ϕ^\perp is just a way of taking into account the existential and single island

aspects of NTL valuations. The same argument used to prove the result for nominals will establish the same result for any existential single island sort, and in particular the same necessary and condition holds for interval nominals.

For languages of interval nominal logic it is easy to ‘read off’ a useful notion of validity preserving morphism without running the danger of collapsing into isomorphism that plagued our investigations in NTL. Let S and T be frames and f a p-morphism from S to T . We say f is *interval invertible* iff for all intervals $I \in \text{Int}(T)$, $f^{-1}[I] \in \text{Int}(S)$. Now we already know that p-morphic interval *models* are equivalent, no matter what sortal constraints are in force, so the following claim should not be surprising: if f is an interval invertible p-morphism from S to T , then $S \models \phi$ implies $T \models \phi$. The point is this; given such a connection between frames then we can always ‘pull back’ any falsifying valuation on T to S , as the inverse images of the denotations of interval nominals are guaranteed to be elements of $\text{Int}(S)$. This establishes the result.

At least one useful interval invertible p-morphism lies to hand: the collapse of Z to $\langle\{0\}, \{\langle 0, 0 \rangle\}\rangle$; the inverse image of the only interval on $\langle\{0\}, \{\langle 0, 0 \rangle\}\rangle$ is the (very large) interval Z . So irreflexivity, asymmetry, and in general m-asymmetry, are not definable in interval nominal languages. As van Benthem points out, antisymmetry is not definable either, as the map $f(n) = n \pmod 2$ from $\langle Z, \leq \rangle$ to the frame with points 0 and 1 bearing the universal relation is an interval invertible p-morphism.

On the other hand, this type of morphism has clear practical and theoretical limits. On the practical side, such morphisms are rather difficult to construct. For example, it seems to be intuitively clear that discreteness is not definable in interval nominal languages, but I do not know how to prove it; the p-morphism that van Benthem [5, pages 160–161] gives to demonstrate this for Priorean languages will not work here, as it is not interval invertible.

More theoretically, I feel that the observation that interval invertible p-morphisms preserve validity tells us rather little about the way interval nominals see frames; it does not help reveal what aspects of frame structure are important to this sort. As with NTL we need new preservation results, but this is very much work in progress.

In the following section we will embed languages with interval nominals into D logic, thus establishing the ultrafilter extension antipreservation result holds for this sort, so

I'll conclude the discussion of the model theory of interval nominals with this remark: many obvious questions demand an answer. In particular: precisely what can be defined with the aid of interval nominals (both on their own, and in collaboration with either or both of variables and nominals), and how do the L_0 expressive capabilities of interval nominals compare with those of purely Priorian and purely nominal languages?

The basic proof theory of this new sort turns out to be straightforward: in particular, the axiomatisation of the minimal logic is reminiscent of K_{nt} , and decidability results are easier to obtain than in NTL.

The minimal logic for languages \mathcal{L} of interval nominal logic can be axiomatised by K_{int} , which is obtained by adjoining to K_t all \mathcal{L} instances of:

$$\begin{aligned} P\epsilon \wedge F\epsilon &\rightarrow \epsilon & (\text{Conv}) \\ \epsilon \wedge E(\epsilon \wedge \phi) &\rightarrow (P\phi \vee \phi \vee F\phi) & (\text{INOM}_W) \end{aligned}$$

Here ϵ is a metavariable over interval nominals. As usual E is a metavariable over existential tenses, and ϕ over arbitrary wffs. The SWEEP_W analog,

$$(\epsilon \wedge \phi) \rightarrow A(\epsilon \rightarrow (P\phi \vee \phi \vee F\phi)) \quad (\text{ISWEEP}_W)$$

could be used in place of INOM_W . We occasionally use the \Diamond_t notation introduced in the previous chapter to abbreviate INOM_W to $\epsilon \wedge E(\epsilon \wedge \phi) \rightarrow \Diamond_t \phi$, and ISWEEP_W to $(\epsilon \wedge \phi) \rightarrow A(\epsilon \rightarrow \Diamond_t \phi)$.

That K_{int} is sound with respect to the class of all frames is clear; to establish strong completeness, argue as follows. Given a K_{int} -consistent set of sentences Σ , form Σ^∞ .⁴ Let $\mathbf{H}^{K_{int}} = \langle H, <_h \rangle$ be the Henkin frame for languages of interval nominal logic. That

⁴We assume the obvious definitions of K_{int} -consistency (which we henceforth refer to as consistency), and related notions in what follows; use the NTL definitions given in Chapter 4 as a guide. Clearly Lindenbaum's Lemma holds for these languages. By Σ^∞ we still mean the Lindenbaum expansion of Σ with respect to some canonical ordering of the wffs of our language.

is, H is the set of all K_{int} -MCSs and $<_h$ is the usual ordering on temporal Henkin frames. Let \mathbf{H}^Σ be the subframe of the Henkin frame $\mathbf{H}^{K_{int}}$ generated by Σ^∞ . The only real task we have to perform is to ensure that for any interval nominal that occurs in some $h \in H^\Sigma$, the set $\{h' \in H^\Sigma : e \in h'\} \in \text{Int}(\mathbf{H}^\Sigma)$, as this establishes that the natural mapping is a weak interval valuation. Let us first show that for all such interval nominals e , $\{h' \in H^\Sigma : e \in h'\} \in \text{Tr}(\mathbf{H}^\Sigma)$. The major burden falls on the INOM_W schema in establishing this.

Suppose there are $h_1, h_2 \in H^\Sigma$ such that $e \in h_1, h_2$, and $h_1 \neq h_2$, $h_1 \not<_h h_2$, and $h_2 \not<_h h_1$. As both h_1 and h_2 are distinct MCS there is a wff that differentiates them; call this wff λ and suppose that $\neg\lambda \in h_1$ and $\lambda \in h_2$. Moreover, as $h_1 \not<_h h_2$ there is a wff ϕ such that $\neg F\phi \in h_1$ and $\phi \in h_2$.⁵ In a similar fashion, as $h_2 \not<_h h_1$ we can find a wff $\psi \in h_1$ such that $\neg F\psi \in h_2$. In short, $\neg\lambda \wedge \neg F\phi \wedge \psi \in h_1$ and $\lambda \wedge \phi \wedge \neg F\psi \in h_2$.

Now as h_1 and h_2 are in the same generated subframe there lies a zig-zag path between them. Hence $E(e \wedge (\lambda \wedge \phi \wedge \neg F\psi)) \in h_1$, where E is the existential tense that accesses h_2 from h_1 via this path. By INOM_W this means that $\Diamond_t(\lambda \wedge \phi \wedge \neg F\psi) \in h_1$.

But $\lambda \wedge \phi \wedge \neg F\psi \notin h_1$, otherwise $\lambda \wedge \neg\lambda \in h_1$, an immediate contradiction. Similarly $F(\lambda \wedge \phi \wedge \neg F\psi) \notin h_1$ as otherwise $F\phi \in h_1$, and hence $F\phi \wedge \neg F\phi \in h_1$. Finally, $P(\lambda \wedge \phi \wedge \neg F\psi) \notin h_1$, as otherwise $P\neg F\psi \in h_1$, that is, $\neg HF\psi \in h_1$. But as $\psi \in h_1$, by TL4 we have that $HF\psi \in h_1$, another contradiction. Thus two such points h_1 and h_2 cannot exist, and for all interval nominals e such that $e \in h$ for some $h \in H^\Sigma$, we have that $\{h' \in H^\Sigma : e \in h'\} \in \text{Tr}(\mathbf{H}^\Sigma)$.

Establishing that this subset is also convex is now easy. Suppose that $h_1, h_2, h_3 \in H^\Sigma$ and that $e \in h_1, e \in h_3$, $h_1 <_h h_2 <_h h_3$ and $e \notin h_2$. But then we have that $Pe \wedge Fe \in h_2$ which by Conv means that $e \in h_2$. Thus we have established that $\{h' \in H^\Sigma : e \in h'\} \in \text{Int}(\mathbf{H}^\Sigma)$.

⁵This follows from the Order Equivalence Lemma of Chapter 4, which among other things states that $h_1 <_h h_2$ iff for all wffs ϕ , $\phi \in h_2$ implies $F\phi \in h_1$. Thus whenever $h_1 \not<_h h_2$ we can find some ϕ such that $\phi \in h_2$ and $F\phi \notin h_1$.

As usual this result only tells us that the natural mapping is a weak valuation — for as in NTL we have that sets of sentences of the form $\{\neg Ei : E \text{ is an existential tense}\}$ are consistent, and such sets ‘drive out’ interval nominals. But as usual we can add a point at infinity, or take disjoint unions, or regenerate starting with a suitable new set of sentences to obtain a valuation. We thus have a interval nominal model, and the usual induction establishes:

Theorem 6.1.1 *K_{int} is strongly complete with respect to I .* □

What about extensions? Firstly the obvious: $K_{int}4$ is strongly complete with respect to the transitive frames, and $K_{int}4T$ with respect to the preordered frames. There is nothing to prove here — as usual the inclusion of 4 guarantees a transitive, and T a reflexive, Henkin frame. However we really should step up and consider the SPOs and POs. The point is this: while many frames do contain *intervals*, not all the frames in I can be said to carry interesting *interval structures*. For a full discussion of interval structures the reader is referred to [5], especially Chapter I.3 (“Periods”), and Chapter I.4 (“Points and Periods”); here we content ourselves with a brief look at van Benthem’s basic definitions. By an *interval structure* or *period structure* is meant a triple $\langle I, \sqsubseteq, < \rangle$. We read \sqsubseteq as the *inclusion relation* on intervals, and demand it be a PO.⁶ We read $<$ as ordinary temporal precedence — essentially the flow of time — and demand it be a SPO. We then put a constraint on how \sqsubseteq and $<$ should interact:

$$\begin{aligned} \forall xy(x < y \rightarrow \forall u \sqsubseteq x(u < y)) & \quad (\text{Left monotonicity}) \\ \forall xy(x < y \rightarrow \forall u \sqsubseteq y(x < u)) & \quad (\text{Right monotonicity}). \end{aligned}$$

It is easy to build interval structures. For example, if we start with an ordinary SPOed frame T and form $Conv(T)$, then $\langle Conv(T), \subseteq, <_{conv} \rangle$ is an interval structure. Here \subseteq

⁶van Benthem further demands that \sqsubseteq satisfies CONJ:

$$\forall xy(xOy \rightarrow \exists z \sqsubseteq x(z \sqsubseteq y \wedge \forall u \sqsubseteq x(u \sqsubseteq y \rightarrow u \sqsubseteq z))).$$

Here O is the predicate ‘overlaps’, defined by $xOy =_{def} \exists u(u \sqsubseteq x \wedge u \sqsubseteq y)$.

is set-theoretic inclusion and $<_{conv}$ is the natural induced precedence relation; this is all discussed on [5, pages 82–83]. For present purposes all we really need to note is that the concept of a frame carrying an interval structure is a much richer one than that of a frame that contains intervals: frames carrying interval structures embody important global structure on the set of intervals — a structure which is necessary if our temporal predikctions are to be honoured in intervallic frameworks. In view of the previously mentioned result, to ensure that our frames possess such pleasant properties we should work on at least the SPOs.⁷

When we consider axiomatisations for the SPOs and POs we obtain sharpening results for both classes of frames, as we can show that simply K_{int4} axiomatises the SPOs, and that K_{int4T} suffices for the POs.⁸

Theorem 6.1.2 *K_{int4} is strongly complete with respect to the class of SPOs.*

Proof:

Given a K_{int4} -consistent set of sentences Σ , expand it to Σ^∞ and form the verifying model $\langle T, V_i \rangle$ in the usual way. Irreflexively bulldoze it, forming $\langle S, V_s \rangle$. What we need to check is that V_s is a interval nominal valuation; that is, for all interval nominals e , $V_s(e) \in Int(S)$. Recall that by the definition of the bulldozing construction given in the previous chapter we have that $s \in V_s(a)$ iff $f(s) \in V_i(a)$, for all atoms a , where f is the bulldozing p-morphism. It is trivial from the bulldozing definition that $V_s(e) \in Tr(S)$. To see that $V_s(e) \in Conv(S)$ for all interval nominals e , we reason as follows. Suppose that $s_1, s_3 \in V_s(e)$ for some interval nominal e , and further suppose that for some $s_2 \in S$, $s_1 <_s s_2 <_s s_3$. As f is a p-morphism it is monotonic in $<_s$, hence $f(s_1) <_t f(s_2) <_t f(s_3)$.

⁷The POs are also pleasant. Of course precedence is reflexive, but we still have all the other intervallic conditions.

⁸Note that all SPOs are in I : all such frames are asymmetric, hence every singleton is an interval.

By assumption $f(s_1)$ and $f(s_3) \in V_i(e)$, but as V_i is an interval nominal valuation, $V_i(e)$ is a convex set, hence $f(s_2) \in V_i(e)$. By the definition of V_s this means that $s_2 \in V_s(e)$, and thus $V_s(e)$ is convex for all interval nominals e .

Thus V_s is an interval nominal valuation, Σ has been verified on a SPO,⁹ and $K_{int}4$ is strongly complete with respect to the SPOs. \square

A similar argument invoking reflexive bulldozing establishes the completeness of $K_{int}4T$ with respect to the POs. Because of these two sharpenings, obtaining completeness results for the classes of frames considered in the previous chapter is routine; we content ourselves with stating the results for STOs and TOs. Firstly, let Tr_ϵ be the schema $P\epsilon \vee \epsilon \vee F\epsilon$; any instance is valid on all transitive frames. The TOs are axiomatised by $K_{int}4T Lin Tr_\epsilon$; and the SPOs by $K_{int}4 Lin Tr_\epsilon$. The results follow by light and heavy bulldozing respectively.

Let us consider whether these logics are decidable. As we can't define irreflexivity and antisymmetry in these languages we might hope that the logics of all the more obvious classes of frames have the finite frame property, and that this can be proved by routine adaptation of filtration arguments. This turns out to be the case.

The definition of filtrations is that given in Chapter 2 for languages of NTL, save that the second two clauses in the definition of V_f now read:¹⁰

$$\begin{aligned} E(t) \in V_f(e) &\text{ iff } t \in V(e), \text{ for all } e \in INOM \cap \Sigma \\ V_f(e) &\in Int(\mathbf{F}), \text{ for all } e \in INOM \setminus \Sigma. \end{aligned}$$

Our ability to form filtrations is unchanged — in particular we can always form Priorean filtrations. However there are two minor difficulties; it is unclear that the V_f obtained

⁹Recall from Chapter 3 our discussion of surjective *model* p-morphisms, in which we commented that no matter what constraints we place on sorts, a p-morphic link between models is all we need to get the usual model equivalence result. The above reasoning establishes that the bulldozing process builds a new interval model p-morphically linked to the original.

¹⁰The notation is that introduced in Chapter 2. The filtration is $\mathbf{M}^f = \langle \mathbf{F}, V_f \rangle$, and Σ is the set of sentences closed under subformulas through which we are filtrating our original model $\langle \mathbf{T}, V_t \rangle$.

after filtrating an arbitrary model $\langle T, V_t \rangle$ through an arbitrary (subformulae closed) set of sentences Σ need be a valuation; further, it is unclear that there need be any intervals at all in a filtration. Once we have solved the first problem, the second yields easily, so we begin by examining the valuations obtained by filtration.

Trichotomy is unproblematic: for all interval nominals e we have that $V_f(e) \subseteq Tr(\mathbf{F})$; this is immediate by the fact that $t <_t t'$ implies $E(t) <_f E(t')$, for all $t, t' \in T$. The problem lies with convexity: how can we guarantee that no point $E(s)$ which falsifies some interval nominal e is placed between two points $E(t)$ and $E(t')$ which verify it?

The solution is straightforward: given Σ , we extend it to a larger (subformulae closed) set of sentences Σ^+ which has the property that any filtration through Σ^+ yields a V_f satisfying the convexity constraint. Let Σ be any set of sentences closed under subformulas. Define:

$$\Sigma' = \{e \rightarrow \neg F(\neg e \wedge Fe) : e \in INOM \cap \Sigma\}.$$

Note that all wffs in Σ' are theorems of K_{int} .¹¹ Let Σ^+ be the result of closing $\Sigma \cup \Sigma'$ under subformulas. We claim that filtrating any interval nominal model $\langle T, V_t \rangle$ through Σ^+ yields a filtration in which every interval nominal $e \in \Sigma$ is assigned an interval in the filtration, because the convexity constraint cannot be violated. Note that in the

¹¹That all such formulas are valid is clear. We can derive instances in K_{int} as follows:

1.	e	Assumption	
2.	$F(\neg e \wedge Fe)$	Assumption	1
3.	GPe	1; TL4, MP	1, 2
4.	$F(\neg e \wedge Fe \wedge Pe)$	2, 3; Inst	1, 2
5.	$F(\neg e \wedge Fe \wedge Pe \wedge e)$	4; Conv, Inst, MP	1, 2
6.	\perp	5; PC	1, 2
7.	$\neg F(\neg e \wedge Fe)$	6; Discharge, PC	1
8.	$e \rightarrow \neg F(\neg e \wedge Fe)$	7; Discharge	

following argument we make use of the Filtration Theorem even though we have not yet established that V_f is an interval valuation. This is perfectly legitimate; it is just as though we were temporarily regarding the interval nominals as ordinary variables, subject to no constraints.¹² That filtration gives rise to an equivalent standard model is not in doubt, and this equivalent standard model verifies formulas containing interval nominals: the point at issue is whether this equivalent model is also an interval nominal model.

So suppose there are $E(t), E(s), E(t') \in \mathbf{F}$ and some interval nominal $e \in \Sigma$ such that $E(t) <_f E(s) <_f E(t')$, $\mathbf{M}^f \models e[E(t)]$, $\mathbf{M}^f \not\models e[E(s)]$, and $\mathbf{M}^f \models e[E(t')]$. This means that in our original model \mathbf{M} we have that $\mathbf{M} \models e[t]$ and $\mathbf{M} \models e[t']$. But $e \rightarrow \neg F(\neg e \wedge Fe)$ is a validity hence it is true at every point in every model, which means in particular that it is true at t in \mathbf{M} , hence by modus ponens we have that $\mathbf{M} \models \neg F(\neg e \wedge Fe)[t]$. By the Filtration Theorem this means that $\mathbf{M}^f \models \neg F(\neg e \wedge Fe)[E(t)]$, as $\neg F(\neg e \wedge Fe) \in \Sigma^+$. On the other hand, as $Fe \in \Sigma^+$, $\mathbf{M}^f \models e[E(t')]$ and $E(s) <_f E(t')$ we have by the second clause in the definition of $<_f$ that $\mathbf{M}^f \models Fe[E(s)]$; and thus $\mathbf{M}^f \models \neg e \wedge Fe[E(s)]$. But $F(\neg e \wedge Fe) \in \Sigma$ and $E(t) <_f E(s)$ so, again by the second clause defining $<_f$ we have that, $\mathbf{M}^f \models F(\neg e \wedge Fe)[E(t)]$ — contradiction.

There remains the second barrier, but it is now clear how to resolve this. If Σ^+ contains even one interval nominal, the previous argument shows that \mathbf{M}^f must contain intervals. So before forming Σ^+ we first enlarge Σ by adding an arbitrary interval nominal to it, and this removes the potential difficulty. The rest is routine. If Σ is a finite set of sentences, Σ^+ is too, thus we can show that the logics of the obvious frames have the finite frame property and are thus decidable; this includes all the logics discussed above. The details are left to the reader.

¹²This is analogous to the strategy used in Chapter 2 to establish the sufficient condition for p-morphisms to preserve nominal validity.

6.2 The shifter

Throughout this thesis it has been said that the introduction of nominals (and now interval nominals) brings important referential mechanisms into tensed languages. While this is true it is slightly misleading, for the task of referring to indices in intensional languages clearly splits into two subtasks. Firstly, we need devices for naming indices — both nominals and interval nominals are examples of this type of device — but secondly, and just as importantly, we need mechanisms which *shift the point of evaluation to the named points*. That is, we need ways of moving to the indicated point as otherwise our names are rather idle additions. This second aspect of referentiality has been ignored till now; the only mechanism we have for performing such shifts are the tense operators. While admirable on many classes of frame — for example, the linear frames omnipresent in natural language semantics — unless the frame geometry is particularly simple they are rather clumsy. In order to use tense operators we have to ‘know the path’ (if there is one) from our present location to the point we wish to access; a ‘jump instruction’ or ‘goto statement’ would be simpler.

We now introduce such an operator: it’s written L and is called the shifter. Semantically it is an ‘everywhere’ — or ‘everywhen’ — operator. That is, $L\phi$ is true iff ϕ is true at all points of the frame.¹³ I call it the shifter because in collaboration with \rightarrow it shifts wonderfully: $L(i \rightarrow \phi)$ jumps to the point named by i and tests the condition ϕ there, and $L(e \rightarrow \phi)$ ‘runs the ϕ test’ on every point in the interval named by e . There is a rather evocative phrase we can borrow from the PDP community: they often talk of ‘content addressable memory’. Suppose we think of a model as a computer memory, with the points of the frame being the memory locations. Accessing these points using tense operators is a little like using a system of pointers and addresses; L , on the other hand, is a ‘content addressable’ operator. $L(\phi \rightarrow \psi)$ checks all the locations holding ϕ to

¹³The idea of using this operator is partly due to Mike McPartlin.

see that they contain ψ as well. More prosaically, we could gloss such conditional forms uses of L by saying ‘whenever ϕ, ψ ’.

We now go on to describe fully referential tensed languages. These will be tensed languages containing both nominal and interval nominals, and in addition the shifter. We develop the languages in two stages. First we add the shifter to languages of NTL, and, once we have described the minimal logic for these languages, go on to add interval nominals as well.

Before going any further I should make it clear that while this work was done independently, every previous person who has considered nominals has introduced the shifter as well. Firstly, Prior discusses tensed languages with shifter and nominals — for example in [79, Appendix B], and [80]. This work will be discussed in the final chapter. Secondly, Robert Bull in [12] introduced a tensed language with strong nominals. His languages are actually a lot more powerful than anything we have considered, as these strong nominals can be quantified over. They also utilise the shifter, and in fact the completeness proof for languages of NTL with shifter sketched below is a subproof of his basic completeness result. Lastly, in a recent manuscript [43], Valentin Goranko and Solomon Passy investigate in detail the effect of introducing the shifter into standard modal languages. In section 8 (‘Several advertisements of the universal modality’) they state:

The prime stimulus for considering the universal modality has come up in the context of the *proper names for the possible worlds* [43, page 22]

They then state how to axiomatise the minimal logic for languages of nominal modal logic with shifter; it’s essentially what’s given below. In short, the idea of the shifter is not new, though I believe the extended completeness results and the decidability results discussed below are original.

We first describe languages of NTL^L . These are languages of NTL augmented by the L operator. Wffs are made in the obvious way, with application of L being allowed in

addition to tense operator application. We define the dual operator M of L by $M\phi =_{def} \neg L\neg\phi$, and read it as ‘Somewhere ϕ ’.

The semantics has already been described; L is a universal ($S5$) operator. We could just add an extra clause to this effect, but completeness proofs proceed a little smoother if we explicitly include the universal relation in the definition of frames for these languages. So, we treat languages of NTL^L in multimodal fashion and say that multiframe for these languages are triples $\mathbf{T} = \langle T, <, U \rangle$ where $<$ as usual is any binary relation on T — it’s this relation that embodies temporal flow — and U is *always* $T \times T$. It’s usually going to be clear from context what our underlying set T is, so we normally drop the subscripts and just write $<$ and U .

We interpret our atoms on these multiframe in the obvious way; valuations can assign atoms arbitrary subsets of the frame, but nominals must be assigned singletons. Tense operators use $<$ as their accessibility relation; and L uses U . That is, $L\phi$ is true at a point t iff ϕ is true at all points U -related to t . As U is $T \times T$ this is clearly what we want.

We now assemble the axiomatisation of the minimal logic in languages of NTL^L . As usual we include K_t . Then, because L is an $S5$ operator, we include the axiomatisation $S5_L$, which consists of K_L ¹⁴ augmented by all instances of $L\phi \rightarrow LL\phi$ (4_L), $L\phi \rightarrow \phi$ (T_L), and $\phi \rightarrow LM\phi$ (B_L). Now we need to add further axioms to control the interactions between L , the tense operators and the nominals. Firstly we add all instances of the following inclusion schema:

$$L\phi \rightarrow H\phi \wedge G\phi \quad (\text{Inc}).$$

Next observe that as every nominal is true at some point in our multiframe, the M operator will always see this point of occurrence; thus we include all instances of:

$$Mn \quad (\text{Force});$$

¹⁴All instances of $L(\phi \rightarrow \psi) \rightarrow (L\phi \rightarrow L\psi)$, together with the rule of necessitation $\vdash \phi \Rightarrow \vdash L\phi$.

recall that n is a metavariable over nominals. Finally we need schemas that guarantee the Unique Occurrence Property. Either of the following variants of our familiar NOM_W and SWEEP_W schemas can be used for this purpose:

$$\begin{aligned} n \wedge M(n \wedge \phi) &\rightarrow \phi && (\text{NOM}_M) \\ (n \wedge \phi) &\rightarrow L(n \rightarrow \phi) && (\text{SWEEP}_L). \end{aligned}$$

In short, $K_{nt}^L = K_t + S5_L + \text{Inc} + \text{Force} + \text{NOM}_M$ (or SWEEP_L). Conspicuous by their absence are schemas such as NOM or SWEEP that explicitly control the interaction of the nominals with the tense operators; in fact they are not needed but are derivable from the interaction of Inc and NOM_M . We now sketch the completeness proof.

Let $\mathbf{H}^{K_{nt}^L} = \langle H, <_h, <_L \rangle$ be the canonical Henkin frame for our language. That is, H is the set of all K_{nt}^L -consistent (henceforth consistent), sets of sentences; $<_h$ is the usual temporal ordering on Henkin frames; and $<_L$ is defined by $h_1 <_L h_2$ iff $L\phi \in h_1$ implies $\phi \in h_2$, for all $h_1, h_2 \in H$.

Given a consistent set of sentences Σ , form Σ^∞ , and let \mathbf{H}^Σ be the subframe of $\mathbf{H}^{K_{nt}^L}$ formed by generating on *both* $<_h$ and $<_L$. That is, \mathbf{H}^Σ is the smallest subframe of $\mathbf{H}^{K_{nt}^L}$ containing Σ^∞ that is closed under both $<_h$ and $<_L$.

First observe that $<_h \subseteq <_L$; this is immediate because of the Inc schema. Next observe that $<_L$ is universal on \mathbf{H}^Σ . This follows from two facts. Firstly \mathbf{H}^Σ was formed by generating on two relations, one of which was a subset of the other. Hence \mathbf{H}^Σ is connected under the larger of the two generating relations, namely $<_L$. Secondly, the schemas 4_L , T_L and B_L guarantee that $<_L$ is an equivalence relation — but any connected equivalence relation is universal. Thus we can justifiably denote $<_L$ by U ($= H^\Sigma \times H^\Sigma$) and write \mathbf{H}^Σ as $\langle H^\Sigma, <_h, U \rangle$; we have the requisite type of multiframe.

The rest is simple. By the Force schema every nominal occurs in at least one $h \in H^\Sigma$; and the familiar style of argument shows that NOM_W guarantees Unique Occurrence. (Note that because $U(= <_L)$ is universal, U -paths are very simple; we can U -step from any $h \in H^\Sigma$ to any other point $h' \in H^\Sigma$ in one stride.) Thus we have the right sort of multiframe, the natural mapping is a valuation, and thus by the usual induction we have:

Theorem 6.2.1 K_{nt}^L is strongly complete with respect to \mathcal{U} . □

Two observations are worth making. The first is that while \mathbf{H}^Σ is U -connected, it need not be $<_h$ -connected.¹⁵ To see this note that the familiar sets of sentences of the form $\{\neg Ei : i \text{ is an existential tense}\}$ are still consistent and will force nominals out from $<_h$ -generated subframes. But such ‘temporally forced out’ nominals are now ‘forced in’ automatically at some other temporal generated subframe by the Force schema; the enrichment of the language has rendered unnecessary our previous strategies for turning weak valuations into ordinary ones.

Secondly, where have our path equations gone? Clearly both NOM_W and SWEEP_W are still valid, so they must be derivable. We show how to derive NOM_W . First observe that from the Inc schema we have that both $\vdash F\phi \rightarrow M\phi$ and $\vdash P\phi \rightarrow M\phi$. But this means that for any n length existential tense we have that $\vdash E\phi \rightarrow M^n\phi$; this follows by induction on n , using the previous two theorems as the base case. But, by the dual of 4_L , we have that $\vdash M^n\phi \rightarrow M\phi$; hence $\vdash E\phi \rightarrow M\phi$. But now we have:

1. i	Assumption	
2. $E(i \wedge \phi)$	Assumption	1
3. $M(i \wedge \phi)$	2; $E\psi \rightarrow M\psi$	1, 2
4. ϕ	1, 3; NOM_M	1, 2
5. $i \wedge E(i \wedge \phi) \rightarrow \phi$	1, 2; Discharge twice	

In short, the path equations were swallowed up by, and are retrievable from, L and M .

Rather than look at extensions of K_{nt}^L we move straight onto languages with shifter, nominals, and interval nominals. More explicitly, we will now work with *languages of temporal reference*, TREF, by which are meant languages with *three* mutually disjoint sorts of atomic symbol, VAR, NOM and INOM, in which wffs are formed by boolean combinations, application of tense operators, and applications of the L operator. The

¹⁵Robert Bull also makes this remark in [12, page 286], attributing the observation to Dov Gabbay.

semantics is defined on multiframes $\mathbf{T} = \langle T, <, U \rangle$ of the kind described for languages of NTL^L , where $\langle T, < \rangle \in I$. Valuations on such multiframes \mathbf{T} are subject to *two* constraints. Firstly, nominals must be assigned singletons; and secondly, interval nominals must be assigned elements of $\text{Int}(\mathbf{T})$.

The minimal logic for such languages is called K_{ref} and it is easily described: it is K_{nt}^L augmented by all instances of Conv , $M\epsilon$ (Force_ϵ), and in addition all instances of:

$$\epsilon \wedge M(\epsilon \wedge \phi) \rightarrow (P\phi \vee \phi \vee F\phi) \quad (\text{INOM}_M)$$

(As usual ϵ is a metavariable across interval nominals. INOM_M is clearly a cousin of INOM_W .)

Completeness is straightforward given what we already know. The first part of the proof merely consists of forming the Henkin frame $\mathbf{H}^{K_{ref}} = \langle H, <_h, <_L \rangle$, where H consists of all the K_{ref} -consistent sets of sentences and $<_h$ and $<_L$ are as described in the completeness proof for K_{nt}^L . Given our consistent set of sentences Σ we take the subframe \mathbf{H}^Σ formed by generating on both $<_h$ and $<_L$, and by the reasoning of the previous proof we see that $<_L$ is universal on \mathbf{H}^Σ . Further, as in the previous proof, the Force schema and the NOM_M schema conspire to force each nominal into exactly one $h \in \mathbf{H}^\Sigma$, and thus the natural mapping satisfies the constraint on assignments to nominals. The only additional work lies in showing that for all interval nominals e , $\{h \in \mathbf{H}^\Sigma : e \in h\} \in \text{Int}(\mathbf{H}^\Sigma)$. As usual, showing this falls into two parts. Firstly, that $\{h \in \mathbf{H}^\Sigma : e \in h\} \in \text{Tr}(\mathbf{H}^\Sigma)$ follows from the INOM_M schema by essentially the same argument that was used for K_{int} in the previous section. That is, we again assume that there are distinct $h, h' \in \mathbf{H}^\Sigma$ containing the same interval nominal and such that $h \not<_h h'$ and $h' \not<_h h$, and manufacture a contradiction as previously shown. The only difference is that we don't use a $<_h$ path to access h from h' and vice-versa, but step directly from one to the other in a single U -step. With this demonstrated, showing that $\{h \in \mathbf{H}^\Sigma : e \in h\} \in \text{Conv}(\mathbf{H}^\Sigma)$ is immediate by the instances of the Conv schema. So we have:

Theorem 6.2.2 K_{ref} is strongly complete with respect to I .

□

Let's turn to extensions. As languages of TREF contain nominals they have a grip on irreflexivity and antisymmetry, so the crucial logics to examine are those of the SPOs and POs. It is natural to suspect that $K_{tref}I4$ (henceforth $I4_{tref}$) and $K_{tref}4TAnti$ (henceforth PO_{tref}), respectively axiomatise these classes. Now clearly, just as with languages of NTL, the Irreflexivity and Simple Cluster Lemmas noted in the last chapter hold. So the way is open to apply bulldozing — that is, if the inclusion of the L operator hasn't affected our ability to bulldoze. In fact it hasn't. This follows from the following general remark about surjective model p-morphisms for languages of TREF.

By a surjective p-morphism f from the multiframe $\mathbf{S} = \langle S, <_s, U_s \rangle$ to $\mathbf{T} = \langle T, <_t, U_t \rangle$ is meant a surjective (temporal) p-morphism f from $\langle S, <_s \rangle$ to $\langle T, <_t \rangle$. The fact that the universal relation U_s is preserved (and anti-preserved) under such mappings is obvious; this holds with any surjection. Next, by a surjective p-morphism f from the TREF model $\mathbf{M}_s = \langle \mathbf{S}, V_s \rangle$ to $\mathbf{M}_t = \langle \mathbf{T}, V_t \rangle$ we mean a surjective p-morphism f between the TREF multiframe \mathbf{S} and \mathbf{T} such that $s \in V_s(a)$ iff $f(s) \in V_t(a)$ for all atoms a . That $\mathbf{M}_s \models \phi[s]$ iff $\mathbf{M}_t \models \phi[f(s)]$ is clear by induction on $deg(\phi)$.

This guarantees that bulldozing — which in this context means the flattening of $<_h$ -clusters in transitive TREF models — is going to work. This immediately yields that $K_{tref}I4$ is complete with respect to the SPOs (by heavy bulldozing), and $K_{tref}4TAnti$ with respect to the POs (by light bulldozing); both results should be clear by considering the completeness proofs for languages of NTL and interval nominal languages on these classes of frames. Note that we move to linear frames the L operator becomes definable in terms of F and P ; L is just \Box_t and M is \Diamond_t . The logic of STOs can be axiomatised by $K_{tref}I4LinTrTr_e$, and that of the TOs by $K_{tref}4TAntiLinTrTr_e$; and the unbounded STOs can be axiomatised by adding all instances of the D schema as well.

As with languages of NTL, many important logics in languages of TREF lack the finite frame property — for example, the logic of any class of frames which validates I, 4, and D. Nonetheless, by exploiting the fact that sorted languages may possess the finite model property while lacking the finite frame property, just as we did with languages of NTL, useful decidability results may be obtained. We examine the case for $I4_{tref}$.

By $\mathcal{M}_{\text{tref}}(I_1)$ and $\mathcal{M}_{\text{tref}}(\text{Tran})$ are meant the classes of TREF models $\langle\langle T, <, U \rangle, V\rangle$ such that $\langle\langle T, < \rangle, V\rangle \in \mathcal{M}(I_1)$ and $\langle\langle T, < \rangle, V\rangle \in \mathcal{M}(\text{Tran})$ respectively; where $\mathcal{M}(I_1)$ and $\mathcal{M}(\text{tran})$ are as defined in the previous chapter. We first observe that $I_{4\text{tref}}$ is strongly complete with respect to $\mathcal{M}_{\text{tref}}(I_1) \cap \mathcal{M}_{\text{tref}}(\text{Tran})$ — this is an immediate corollary of the completeness proof for $I_{4\text{tref}}$, as the Henkin model there generated is in this class — and now seek to show by means of a filtration argument that $I_{4\text{tref}}$ has the finite model property with respect to this class.

As we are working in a multimodal framework with two distinct referential sorts, we are going to have to augment our filtration technique somewhat. So, given a TREF model $\mathbf{M} = \langle\langle T, <, U \rangle, V\rangle$ and a finite subformulae closed set of sentences Σ , we first define F to be $\{E(t) : t \in T\}$, where, as in Chapter 1, the $E(t)$ are equivalence classes of T created by Σ agreement. How do we proceed from here?

We define our filtration \mathbf{M}^f of \mathbf{M} through Σ to be $\langle\langle F, <_f, U_f \rangle, V_f\rangle$, where $\langle\langle F, <_f \rangle, V_f\rangle$ is the ordinary Priorean filtration of \mathbf{M} through Σ we are used to, and U_f is just $F \times F$. The important thing to note is that $\langle\langle F, U_f \rangle, V_f\rangle$ is the smallest filtration of $\langle\langle T, U \rangle, V\rangle$ through Σ . This is immediate: it follows from the universality of U that sUt iff $E(s)U_f E(t)$, which is the way smallest filtrations are defined.¹⁶ But this means that the structure \mathbf{M}^f we have defined is just a componentwise filtration of both relations on \mathbf{M} through Σ . This guarantees that

$$\forall \sigma \in \Sigma (\mathbf{M} \models \sigma[t] \text{ iff } \mathbf{M}^f \models \sigma[E(t)]),$$

by Segerberg's statement of the Filtration Theorem [97, page 303], which is stated in full multimodal generality for all such 'componentwise filtrated' constructions.

In short, defining filtrations for tensed languages with an additional universal operator is straightforward. It only remains to ensure that we end up with the desired relational structure on our filtrations. As we took a Priorean filtration to make $<_f$, and as we

¹⁶In fact this filtration is the *only* filtration of \mathbf{M} through Σ that exists — the largest, the smallest, and hence all other filtrations coincide if we start with a universally related model.

directly built in a universal relation on F , what this amounts to is that we must guarantee that the filtration contains irreflexive points, that all nominals denote these, and that interval nominals denote intervals. We know from our previous discussions that all this can be achieved by judiciously extending our original subformulae closed set of sentences Σ to a new such set Σ^+ . That is, given Σ , if Σ contains no nominals we add an arbitrary nominal i to Σ ; and if it contains no interval nominals we add an arbitrary interval nominal e . Call this (possibly new) set Σ' . We then define Σ'' to be:

$$\Sigma' \cup \{i \rightarrow \neg Fi : i \text{ occurs in } \Sigma'\} \cup \{e \rightarrow \neg F(\neg e \wedge Fe) : e \text{ occurs in } \Sigma'\}.$$

Finally, we let Σ^+ be the smallest subformula closed set of sentences containing Σ'' . By the reasoning of the previous chapter and the previous section, these additions have the desired effect. Thus \mathbf{M}^f is a finite member of $\mathcal{M}_{tref}(I_1) \cap \mathcal{M}_{tref}(Tran)$, and the usual argument shows that $I4_{tref}$ is decidable.

Other extensions in this vein are straightforward. In particular, we can show that PO_{tref} has the finite model property with respect to $\mathcal{M}_{tref}(SC_1) \cap \mathcal{M}_{tref}(Pre)$, where these classes of models are defined in the obvious way from their NTL correlates. The proof is essentially that just given, save that when we define Σ'' we add all Σ' relevant instances of $i \rightarrow G(Fi \rightarrow i)$, the antisymmetry axiom, instead of the relevant instances of the irreflexivity axiom. The reasoning of the previous chapter again establishes that this has the required effect.

One of the motivations for developing NTL and the other sorted languages described here has been the wish to model at least the more obvious facts of natural language temporal reference and its interaction with tense in constrained languages. The demonstration that this can be done must wait till the next chapter, but we should pause to check that languages of TREF can still be described as constrained; as van Benthem has shown in great detail,¹⁷ intuitive tense logics are often augmented with extra machinery to the point where they become (unadmitted) notational variants of first order languages. Where do we stand with TREF?

¹⁷There is some discussion of this in [5, pages 132–133], and it's the major point of [4].

We already know that the addition of nominals and interval nominals increases the expressive power of tensed languages; what about the shifter? This addition takes us still higher: to select a particularly simple example from the work of Goranko and Passy [43, page 11], we can now define $\exists t_1 t_2 (t_1 < t_2)$; that is, the class of frames such that $< \neq \emptyset$. Simply $MF\top$ accomplishes this. Note that this class is not definable in a tensed language — even with the help of nominals or interval nominals — because of the generated subframe validity preservation result. However we can still claim to be working in a constrained class of languages, as languages of TREF can be simulated by tense logic augmented by the D operator, as will now be shown.

We have already seen that nominals can be simulated by means of the uniqueness operator U that is definable in terms of D . Further, as we saw in Chapter 2 (where Ron Koyman's A notation was used), the shifter can be defined in terms of D : $L\phi =_{def} \phi \wedge \neg D\neg\phi$. The only messy part is simulating interval nominals. For this purpose we need a defined operator which we dub I :

$$I\phi =_{def} (P\phi \wedge F\phi \rightarrow \phi) \wedge (\phi \wedge D\phi \rightarrow F\phi \vee P\phi) \wedge M\phi.$$

The first conjunct is Con ; the last Force ; the second uses D to force trichotomous behaviour.

So, given any TREF formula $\phi(i_1, \dots, i_n, e_1, \dots, e_m)$, where the subscripted i s and e s are all and only the occurring nominals and interval nominals in ϕ , we can construct a logically equivalent formula using just variables, F , P , and D as follows. Let $\phi^D(i_1, \dots, i_n, e_1, \dots, e_m)$, be the result of replacing all L s in ϕ using the equivalence $L\phi = \phi \wedge \neg D\neg\phi$. Then we have that for any (temporal) frame $\mathbf{T} = \langle T, < \rangle$ that:

$$\langle \mathbf{T}, U_t \rangle \models \phi(i_1, \dots, i_n, e_1, \dots, e_m)$$

if and only if:

$$\langle \mathbf{T}, \neq_t \rangle \models Up_1 \wedge \dots \wedge Up_n \wedge Iq_1 \wedge \dots \wedge Iq_m \rightarrow \phi^L(p_1/i_1, \dots, q_m/e_m);$$

where the $p_1, \dots, p_m, q_1, \dots, q_m$ are variables not occurring in ϕ . This translation gives us an useful upper bound on the expressive power of TREF. In passing note that it also

shows that TREF validity is antipreserved under the formation of ultrafilter extensions; we need merely translate and appeal to Maarten de Rijke's result in [89, page 6].

This concludes the presentation of languages of TREF. We will further develop them in the following chapter to allow them to model common features of temporal reference and their interaction with tense. However a further logical topic beckons: giving a more general account of sorting, and we devote the following section to this.

6.3 Sorting generalised

We have encountered several sorts in this thesis: nominals, weak nominals, interval nominals, and the 'separated sort' used in Chapter 4 to show that $K + \text{NOM}_W$ does not axiomatise the minimal nominal modal logic. It is easy to give other examples: sorts constrained to denote precisely 2, 3 or m distinct points; sorts constrained to denote a finite set of points, or a countable infinity of points; or a sort that denotes equivalence relations. However we will not extend the catalog of sorts in this chapter; rather the discussion will revolve around two questions: does the imposition of sorting have any effect on what should be regarded as logically paramount, and might it be possible to develop any general theory of sorted intensional logics? Briefly, my answer to the first question will be 'Yes', and my answer to the second 'I don't know'. The positive answer to the first question comes from considering what a sorted intensional logic might be, and noting the impact the failure of substitutivity has. The answer to the second question is my response to the variety and complexity of issues raised by sorting. I do have one positive suggestion however: alternative semantics for our languages should be explored if these wider issues are to be addressed, and I briefly sketch two alternatives.

Intuitively a sorted intensional language is an intensional language whose atoms are divided into mutually disjoint sets, or sorts. Associated with each sort is a constraint on the functions that count as valuations. This is vague: it is not clear what sort of constraints can be placed on valuations. Are global demands, of the type made by strong languages of NTL to be permitted? I have no wish to rule out such possibilities —

as the example of strong NTL shows, such constraints can be very important — but for the purposes of this chapter sorting will have a narrower sense: only local constraints on valuations will be considered. Let's define this precisely.

Given a non-empty set Δ , by an (uninterpreted) Δ -sorted language of tense logic \mathcal{L}^Δ is meant a pair $\langle LOG, \{S_\delta : \delta \in \Delta\} \rangle$, where LOG is as defined in Chapter 2; for each $\delta \in \Delta$, S_δ is countably infinite; for all $\delta, \delta' \in \Delta$, $\delta \neq \delta'$ implies $S_\delta \cap S_{\delta'} = \emptyset$; and $LOG \cap \bigcup \{S_\delta : \delta \in \Delta\} = \emptyset$. We say that such a language has syntactic signature Δ . By the atoms of \mathcal{L}^Δ , $ATOM_{\mathcal{L}^\Delta}$, is meant $\bigcup \{S_\delta : \delta \in \Delta\}$; and an atom that is an element of S_δ is called an atom of sort δ . The wffs of \mathcal{L}^Δ are made in the usual tense logical fashion out of these atoms using the elements of LOG as connectives. Thus syntactically the wffs of a Δ -sorted language \mathcal{L}^Δ look like ordinary tense logical wffs, save that at the atomic level we need to determine not only that a non-logical symbol is a symbol of the language, but what sort it is as well. For practical purposes we will only be interested in Δ -sorted languages where Δ is recursive, indeed finite.

At the heart of the semantics for these languages are Δ -sorted frames. A Δ -sorted frame is a triple $\mathbf{T}^\Delta = \langle \mathbf{T}, \Delta, f \rangle$ where \mathbf{T} is a frame, Δ a non-empty set, and $f : \Delta \rightarrow Pow(Pow(T))$. We say such a frame has signature Δ . Given a fixed Δ , any Δ -sorted language \mathcal{L}^Δ must be interpreted on a Δ -sorted frame. By a valuation V for such a language on such a frame $\langle \mathbf{T}, \Delta, f \rangle$ is meant a function from $ATOM_{\mathcal{L}^\Delta}$ to $Pow(T)$ such that for all $a \in ATOM_{\mathcal{L}^\Delta}$, $a \in S_\delta$ implies $V(a) \in f(\delta)$. We extend V to a function defined on all wffs in the usual fashion. If $\mathbf{T}^\Delta = \langle \mathbf{T}, \Delta, f \rangle$ is a Δ sorted frame, and V a valuation for \mathcal{L}^Δ on \mathbf{T}^Δ , we call $\mathbf{M}^\Delta = \langle \mathbf{T}^\Delta, V \rangle$ a Δ -sorted model; and for any wff ϕ of our language, and any $t \in T$, we say that $\mathbf{M}^\Delta \models \phi[t]$ iff $\langle \mathbf{T}, V \rangle \models \phi[t]$. This last definition makes use of the fact that the frames \mathbf{T} underlying Δ -sorted frames \mathbf{T}^Δ are just ordinary frames, and the valuations on them are just ordinary valuations: nothing has changed save that we have sortal constraints at the atomic level. The extra machinery is simply to allow such constraints to be expressed. ¹⁸

¹⁸Another piece of machinery could prove useful: splitting valuations V into Δ -indexed sub-

As an example of these definitions consider mixed languages of NTL. We can regard them as Δ -sorted language where Δ is any doubleton. Let 2 be the doubleton $\{0, 1\}$, and let us express what it is for a 2-sorted language \mathcal{L}^2 to be a mixed language of NTL. The intended interpretation for mixed languages of NTL is that variables denote arbitrary subsets of arbitrary frames, and nominals singleton subsets of arbitrary frames. Thus if a 2-sorted language is to be a mixed language of NTL we must interpret it on 2-sorted frames of the form: $\langle T, 2, \{\langle 0, X \rangle, \langle 1, Y \rangle\} \rangle$, where $T \in \mathcal{U}$, $X = Pow(T)$, and $Y \in Pow(Pow(T))$ such that $y \in Y$ iff y is a singleton subset of T . The talk of ‘intended interpretation’ should make it clear that we are going to have to relativise the interpretation of Δ -sorted languages to particular classes of Δ -sorted frames if we are to capture the particular constraints we are interested in. For example, there is no syntactic difference between mixed languages of NTL, mixed languages of weak NTL, and languages with variables and interval nominals: all three languages are just 2-sorted languages, and the difference between them lies in their intended interpretation. Thus to capture this we now define what interpreted Δ -sorted interpreted languages are, and give a notion of validity.

Given a fixed non-empty set Δ , by a Δ -sorted *interpretational base* is meant a non-empty collection \mathcal{T}^Δ such that each element in \mathcal{T}^Δ is a Δ -sorted frame. (That is, we gather together a collection of sorted frames of the same signature.) Given a Δ -sorted language \mathcal{L}^Δ , and any Δ -sorted interpretational base \mathcal{T}^Δ , valuations exist for \mathcal{L}^Δ on each element of \mathcal{T}^Δ , as the syntactic and frame signatures match, and thus we can interpret our language on any element of the interpretational base. We call such a pair $\langle \mathcal{L}^\Delta, \mathcal{T}^\Delta \rangle$ an *interpreted Δ -sorted language*, or simply an interpreted language. This notion of interpreted languages yields the required distinctions between sorted languages of the same signature. For example, let \mathcal{L}^2 be any 2-sorted language. Then the interpreted

valuations V_δ such that $V_\delta : S_\delta \rightarrow f(\delta)$. This is probably a neater way to define things, but we won’t need it in what follows.

language $\langle \mathcal{L}^2, \mathcal{T}_I^2 \rangle$, where \mathcal{T}_I^2 is

$$\{ \langle \mathbf{T}, 2, f \rangle : \mathbf{T} \in I \text{ and } f(0) = \text{Pow}(\text{Pow}(T)) \text{ and } f(1) = \text{Int}(\mathbf{T}) \}$$

is what we mean by a language of interval nominal logic.

With this to hand we can now define a notion of validity. We say $\mathcal{T}^\Delta \models \phi$ iff for all $\mathbf{T}^\Delta \in \mathcal{T}^\Delta$, all valuations V for \mathcal{L}^Δ on \mathbf{T}^Δ , and all $t \in T$, $\langle \mathbf{T}^\Delta, V \rangle \models \phi[t]$.

It is straightforward to extend these definitions to multimodal systems — essentially this amounts to changing the definition of LOG to $(LOG \setminus \{F, P\}) \cup \{\Diamond_\lambda : \lambda \in \Delta\}$, where Δ is some new index set. As the concept of an ‘intensional language’ is somewhat elastic, it is pointless to try and give a fully general definition of a sorted intensional language, but the basic idea should be clear, and can be adapted to other intensional systems. In the following chapter, for example, we use sorting in a double indexed system.

Given a sorted language \mathcal{L}^Δ , what subsets of $WFF_{\mathcal{L}^\Delta}$ should count as intensional logics in \mathcal{L}^Δ ? At the beginning of Chapter 4 we stipulated that tense logics were sets of formulas closed under modus ponens and temporal generalisation. A standard demand was omitted from this specification: the demand that tense logics be closed under substitution.

The reasons for omitting this demand are clear: the theories of our structures should be logics, but such theories are not substitution closed. For example, $i \rightarrow \neg Fi$ is valid on the SPOs, but $p \rightarrow \neg Fp$ is not. Looking at the matter axiomatically, an unrestricted rule of substitution is not a sound rule — we don’t want to derive $Pp \wedge Fp \rightarrow p$ in K_{int} , for example. This failure of substitutivity is neither surprising nor objectionable — indeed we might say that the failure of substitutivity is the hallmark of sorted systems. Underlying sorting is the idea that different types of information are combined in uniform fashion. Given that the types of information are genuinely different, this difference must emerge somewhere, and it does so at the level of logics. On a fixed class of frames, different schemas may be validated for the different sorts: sorted sublogics arise.

Nonetheless, while the failure of substitutivity is to be expected, indeed desired, the ramifications of its failure are many and often surprising. Probably the deepest we have

so far encountered is the failure of Segerberg's Theorem in NTL: sorted frameworks can have the finite model property even though they lack the finite frame property. Another consequence, which nicely illustrates the care we must exercise when dealing with sorted languages, concerns the conditions nominalisations of purely Priorean formulas define. Suppose ϕ_1 and ϕ_2 are two Priorean formulas that define the same class of frames \mathcal{T} . Let ϕ_1^n be the nominalisation of ϕ_1 , and ϕ_2^n be the nominalisation of ϕ_2 , let \mathcal{T}_1 be the class of frames defined by ϕ_1^n , and \mathcal{T}_2 be that defined by ϕ_2^n . Although by the Inheritance Lemma we have that $\mathcal{T} \subseteq \mathcal{T}_1$ and $\mathcal{T} \subseteq \mathcal{T}_2$, it need not be the case that $\mathcal{T}_1 = \mathcal{T}_2$. For example, we have already seen that the nominalisation of Löb's formula, $G(Gi \rightarrow i) \rightarrow Gi$ defines the condition

$$\forall i \forall t_0 (\forall x (t_0 < x \rightarrow (\forall y (x < y \rightarrow i = y) \rightarrow i = x) \rightarrow \forall z (t_0 < z \rightarrow z = i)).$$

Note that this condition holds of the singleton reflexive loop $\langle \{0\}, \{(0, 0)\} \rangle$, and hence we know that $G(Gi \rightarrow i) \rightarrow Gi$ is valid on this frame. (In passing, Löb's formula itself is not valid on this frame, nor on any frame containing reflexive loops.) Now consider the variant of Löb's formula obtained by uniformly substituting $\neg p$ for p , $G(G\neg p \rightarrow \neg p) \rightarrow G\neg p$. This formula defines the same class of frames as the original —

— however it's nominalisation $G(G\neg i \rightarrow \neg i) \rightarrow G\neg i$ defines irreflexivity. This can easily be seen by considering its contrapositive, $Fi \rightarrow F(i \wedge \neg Fi)$.

Simpler examples abound. Consider p . This defines the empty class of frames, as does $\neg p$. But the nominalisation of p is i , which defines the class of singleton frames, and the nominalisation of $\neg p$ is $\neg i$, which defines the empty class of frames. In both this example and the case of Löb's formula, the negation has performed a 'sortal transformation'. Nominals are the sort 'just above' \perp , and negating them produces a 'composite' or 'defined' sort 'just below' \top . There is no reason to suppose that two such different sorts will exhibit the same logical behaviour — and as these examples show, they don't.

Let us turn from these model theoretic questions to completeness theory, and our original question: what should count as a sorted intensional logic? We begin by observing

that one of the effects of the failure of substitutivity is to render frame incompleteness results a virtual triviality.

We know that $K_{nt} + (FF\phi \rightarrow F\phi)$ is sound and complete with respect to the class of transitive frames; the added 4 schema insists that arbitrary information is transitively accessible. But can we be more parsimonious in our axiomatisation; is $K_{nt} + (FFn \rightarrow Fn)$ — or 4^n — also complete with respect to the transitive frames? Intuitively no. Adding just this schema will not give us enough information about the Henkin frame to yield the desired result, anymore than adding $i \rightarrow \neg Fi$ to K_{nt} forces irreflexivity on the Henkin frame. In fact the axiomatisation $K_{nt} + (FFn \rightarrow Fn)$ is not only incomplete with respect to the class of transitive frames, it is not characterised by any class of frames at all. This can be seen as follows. First a general note. Let $K_{nt}\Sigma$ be K_{nt} augmented by some collection of axioms Σ . Suppose \mathbf{M} is a model such that for all $\sigma \in \Sigma$, $\mathbf{M} \models \sigma$. Then we have that $\vdash_{\Sigma} \phi$ implies $\mathbf{M} \models \phi$. This is merely soundness stated generally, and follows because modus ponens and necessitation preserve truth in a model.

Now let \mathbf{T} be the frame $\langle \{1, 2, 3\}, \{\langle 1, 2 \rangle, \langle 2, 3 \rangle\} \rangle$. Although \mathbf{T} is an intransitive frame, it is easy to make all nominal instances of the transitivity schema true in this model, while falsifying all variable instances of the schema. For consider the valuation V that assigns all nominals $\{i\}$ and all variables $\{3\}$. Clearly all the axioms are true in this model, and equally clearly all variable instances of the transitivity schema are false at 1. Hence for any variable p , $\not\models_{4^n} FFp \rightarrow Fp$. Moreover I_n cannot be characterised by any class of frames at all, for any characterising class would have to be transitive, and hence would validate $FFp \rightarrow Fp$ which is not derivable. In short, K_{nt} is not strong enough to transfer the ‘transitive accessibility’ of singleton information to arbitrary information.¹⁹

¹⁹On the other hand, if we were working with a language of strong nominals, adding the 4 schema for nominals only would suffice. In such a language, given that we knew the truth of $FF\phi$, any ϕ verifying time must be ‘marked’ by some strong nominal, say i , by strongness. Hence we could deduce $FF(i \wedge \phi)$. But this means that FFi is true, and from the axiom $FFi \rightarrow Fi$ we deduce Fi . But then from $Fi \wedge FF(i \wedge \phi)$ we obtain $F(i \wedge \phi)$ by *NOM*, and hence $F\phi$. In

Frame incompleteness results for purely Priorean languages are not such trivialities — indeed the discovery of such results, and ^{the} second order perspective on intensional logic which attended them, marked the beginning of a new and important phase in intensional logic — and the ease with which we obtained the result above may suggest that nothing is at stake here save the correct definition of what a sorted logic should be. Perhaps a natural closure condition on logics is missing — some sortal approximation to substitutivity — which would rule out 4^n as a logic? This may well be, and it is true that 4^n does not seem particularly interesting; but there are natural frame incomplete logics, where the incompleteness, as with 4^n , trades on sorting and can be proved with similar ease. Such logics are harder to dismiss as pathologies.

Consider the TREF axiomatisation $I4_{tr\epsilon}LinTrTr_{\epsilon}D$, which for the remainder of this discussion we will call $LIN_{\epsilon}D$. As we have already noted, this logic is strongly complete with respect to the unbounded STOs. Now consider the extension of $LIN_{\epsilon}D$ obtained by adjoining as axioms all instances of

$$\epsilon \rightarrow (F\neg\epsilon \wedge P\neg\epsilon).$$

At first glance this may appear absurd. We know that on the transitive frames any instance of $\epsilon \rightarrow F\neg\epsilon$ defines the condition

$$\forall x\exists yz(x < y \wedge x < z \wedge \neg Tri(y, z)),$$

and clearly any instance of $\epsilon \rightarrow P\neg\epsilon$ defines the mirror image condition. As the $LIN_{\epsilon}D$ axioms demand that time is an unbounded STO, accepting the new axioms seems tantamount to demanding the existence of a STO containing non-trichotomous pairs of points. However this is illusory. The new axiomatisation is not inconsistent. It *is* frame incomplete, but nonetheless it is sound and complete with respect to a rather natural class of *models*.

short, because strong valuations cover frames, the transitivity stipulation transfers from strong nominals to arbitrary information.

The point is that we need not interpret the truth of the new axioms as reflecting a constraint on frame orderings. We can view matters rather more simply: the new axiom reflects a (possible) truth concerning our usage of the word ‘interval’: that intervals must be ‘small’. The new axiom asserts that intervals cannot be insanely large: real intervals have a beginning and an end, and our adoption of this axiom reflects in our temporal logic this understanding of what it is to be an interval in our logic. Before discussing the underlying intuitions further, let’s prove the soundness and completeness results.

Call the augmented axiomatisation $LIN, DSmall$. Let \mathcal{V}_{USTO} denote the class of TREF valuations for our language on the class of unbounded STOed TREF frames, and let \mathcal{V}_{SMALL} be the (non-empty) subclass of \mathcal{V}_{USTO} defined as follows. A valuation $V \in \mathcal{V}_{USTO}$ on $\mathbf{T} = \langle T, <, T \times T \rangle$ is in \mathcal{V}_{SMALL} iff for all interval nominals e , there exist an $u \in T$ such that for all $t \in V(e)$, $t < u$; and for all interval nominals e , there exist an $l \in T$ such that for all $t \in V(e)$, $l < t$. That is, we are interested in the class consisting of all valuations on unbounded STOs such that assignments for all interval nominals e are bounded above and below. The result we seek is that $LIN, DSmall$ is sound and complete with respect to all models $\langle \mathbf{T}, V \rangle$ such that \mathbf{T} is an unbounded STO, and $V \in \mathcal{V}_{SMALL}$.

Soundness is straightforward. As $\mathcal{V}_{USTO} \subset \mathcal{V}_{SMALL}$ the soundness of the LIN, D axioms is assured. The only thing that requires checking is that the instances of *Small* are true in all such models, but the restriction to the valuations in \mathcal{V}_{SMALL} is precisely what is required to achieve this. All the rules of inference preserve truth in a model, thus soundness is proved. This in turn means that the logic is consistent, hence it is not characterised by the empty class of frames. As this was the only possible class of frames which could have characterised it, the logic cannot be characterised by any class of frames at all and is frame incomplete.

To obtain the indicated completeness result, we reason as follows. Given a consistent set of sentences, we make the Henkin frame in the usual way. Because of the presence of *Lin*, *Tr*, and *Tr_ε*, we know that the generated subframe is ‘almost linear’ (it may contain clusters), that it is unbounded because of the *D* schema, and that every nominal

and interval nominal occurs in some MCS. We know we can heavily bulldoze this frame into an unbounded STO, but before we do so let's check that the natural valuation is bounded above and below. That is, we want to show that it is impossible that any interval nominal e can be in some point h in the Henkin frame, and also in all h' such that $h <_h h'$, or all h' such that $h' <_h h$. So suppose there is some interval nominal e such that $e \in h$, and for all h' such that $h <_h h'$, $e \in h'$. This means that $Ge \in h$, or $\neg F\neg e \in h$. But all instances of $e \rightarrow F\neg e \in h$, hence as $e \in h$, $F\neg e \in h$ — contradiction. Showing that the natural valuation is bounded below is similar. Given this, and recalling that the set of points in the generated Henkin model containing any interval nominal e must be trichotomous and convex, it is clear that heavily bulldozing the frame preserves the 'boundedness' property of the natural valuation. But as the bulldozed frame is an unbounded STO, its valuation is in \mathcal{V}_{SMALL} , and we have a model in the desired class and our completeness result.

A second example of this type of logic is LIN, D augmented by all instances of $\epsilon \rightarrow (P\epsilon \vee F\epsilon)$. Read as reflecting a possible constraint on the entities we can call intervals, the axiom is perfectly reasonable. It says that intervals must take time; points are not to be considered intervals as they are too small — real intervals take time. In view of this we call the schema *Dur*, for duration.

Call the nonempty class of valuations on the unbounded STOed TREF frames such that for all interval nominals e , $card(V(e)) > 1$, \mathcal{V}_{Dur} . It is easy to show that $LIN, DDur$ is sound and complete with respect to the class of all TREF models $\langle T, V \rangle$ such that T is an unbounded STOed TREF frame, and $V \in \mathcal{V}_{DUR}$.

Now I am not particularly concerned whether these three examples are 'universal truths of usage' or 'natural language conventions' concerning which stretches of time we can properly dub intervals or not. The *Dur* schema appears problematic — making a decision either way seems arbitrary. The *Small* schema is perhaps more plausible: there certainly are some clear examples of interval names which do obey this constraint. For example, once only dates — 12th July 1959 — do, as do 'periodically repeating' interval names such as Monday. Furthermore — and this is one of the joys of sorting — no claim

of universality need be made to defend the use of this axiom. Given a clear example of unbounded interval names — we might dub these ‘scientific’ or ‘metaphysical’ interval names, given their likely source — we need merely subdivide our interval nominals into two subsorts — a bounded and an unbounded sort — and only add the instances of this schema in the bounded sort. (And clearly we *must* add the instances of this schema, or some equivalent, in such a sort. This bounded behaviour constitutes part of the logical properties of such names, and we cannot hope to accurately model their behaviour without imposing such a restriction.) Nonetheless, I don’t think either axiom is particularly exciting; what I find more interesting are the ideas these logics point towards: the importance of classes of models, rather than classes of frames, for applied temporal logic; or, more accurately, the importance of considering systematic constraints on valuations which are syntactically reflected in our tensed object languages.

Applying temporal logic is about modeling. At the very least what we are doing is attempting to reflect, to a degree of accuracy suitable for the purposes at hand, some aspects of the problem (physical, linguistic or computational) in the semantic structures of our languages. Given some problem we seek to provide a ‘picture’ of it, an image that is not too simplified, in the semantic structures our logical languages talk about. Then, by examining the logics of these ‘pictures’, we hope to provide useful answers to our questions.

But what tense logical semantical structures should be used, frames or models? The tense logical tradition has revolved around ‘photographing intuitive temporal ontology’ in frames.²⁰ Now up to a point this is sensible: it is clearly crucial to know what logics our modeling choices regarding the flow of time give rise to. However implicit in any position which regards only the relational structure of frames as logically important is the claim that the flow of time constitutes our temporal ontology, and this I believe is mistaken. Equally important are our pretheoretic notions concerning events, and event

²⁰As we shall see in the final chapter, constraints on valuations have often been imposed in interval based logics.

structure — in short, the intuitive constraints we feel are placed on the distribution of information on the temporal flow. Given that such constraints are important, and given that different types of information are subject to different constraints, we are led to a position which demands that we investigate not just the effects of the varying relational structure, but in addition the effects of sortally constraining valuations as well.

The logic $LIN, DSmall$ illustrates, in rather simplistic fashion, the results of attempting this sort of modeling. The example in certain respects is rather artificial; rather than reflecting some aspect of natural ‘event structure’, the *Small* axiom reflects a certain, rather peripheral, aspect of referential usage. (In general, all uses of sorting to model reference have this ‘conventional’ aspect. In many ways this is less exciting than modeling event structure, but on the other hand some pleasant results can be achieved in such modeling, and the problems are by no means always peripheral, as we shall see in the next chapter.) Nonetheless, this insistence that intervals be bounded is a useful illustration of a general point. We tried to model a certain convention regarding referential usage, and succeeded in doing so in a natural way: the ‘pictures of time’ with respect to which we managed to prove a completeness result ‘looked right’: the flow was the desired one (an unbounded STO), and correctly decorated with the right sort of referential information. In short the model we built, in which both the temporal flow and the (referential) information distribution were correctly constrained reflected our intuitions. Such logics should not be regarded as odd just because they cannot be captured by constraints on frame ordering: there is no particular reason why we should want to so capture many sorted logics in the first place. Such completeness results — and their associated completeness results — are the natural concomitants of the view of temporal modeling here expounded.

I am not going to discuss this matter further. (This is not because I believe there is nothing more to be said, but because an adequate treatment is impossible here.) I’ll simply state that I believe the discussion so far has established the importance of classes of models, rather than classes of frames, as our fundamental semantic touchstone. This granted, I will now consider other matters. In particular, I want to argue that if we are to achieve any general results concerning sorting we should consider alternative semantics

for our languages. As a first step, let's consider the problems that will be faced by Δ -sorted languages, and what we might hope from a general theory of sorting.

Sorting any intensional language raises anew the traditional problems of definability, completeness and decidability. Moreover, although prior acquaintance with other sorts may prove useful, solving these problems is a new exercise each time, sometimes a difficult one. Consider sorting a standard tensed language by introducing the sort of 2-nominals i_2, i'_2, i'_3, \dots , the sort constrained to denote precisely 2 distinct points. How can we axiomatise its minimal logic on the basis of K_t ? Certainly K_t itself won't do:

$$i_2 \wedge p \wedge Fi_2 \wedge \neg Fp \rightarrow (E(i_2 \wedge \phi) \rightarrow (\phi \vee F\phi))$$

is valid, whereas

$$q \wedge p \wedge Fq \wedge \neg Fp \rightarrow (E(q \wedge \phi) \rightarrow (\phi \vee F\phi))$$

is not. The solution promises to be messy, yet this is a fairly obvious variant of NTL; for more exotic sorts the problems may be severe.²¹

Moreover, sorting does not only inherit the old problems, it gives rise to new problems of its own. A good example concerns *transference between sorts*, something briefly touched on when discussing the incompleteness of 4^n . Consider a mixed language of NTL. Are there examples of logics which we can axiomatise by adding only schemas in nominals, but which are *not* conservative over K_{nt} with respect to the purely Priorean formulas provable? That is, are there axiomatisations $K_{nt}\Sigma$ where all the axioms in Σ are

²¹Actually, although 2-nominals seem similar to nominals, they differ in an important respect: nominals are a single island sort, whereas 2-nominals are not. This is the source of the messiness in axiomatising this sort. In passing, for any cardinal number $c \geq 1$ we could define *c*-gsf-nominals: a sort constrained to denote precisely *c* distinct points, no two of which can lie in the same generated subframe. For all $c \geq 1$ the minimal tense logic for purely *c*-gsf-nominal languages is axiomatised by K_{nt} . To see this, simply doubly generate an NTL model in the usual way (we can do this because of the NOM_W schema), and then add the required number of isolated points at infinity to provide the remaining references for the *c*-gsf-nominals.

purely nominal wffs, and yet there exists a purely Priorean formula ϕ such that $\vdash_{K_{nt}\Sigma} \phi$ and $\not\vdash_{K_{nt}} \phi$? We have seen no examples of such a phenomenon so far: the axiomatisation I we considered in the previous chapter, for example, though of the requisite form was conservative over K_{nt} in this fashion. Nonetheless, simple examples lie to hand

Firstly consider the axiomatisation $K_{nt} + n$. This is complete with respect to the class of all singleton frames. Proving this could hardly be simpler: any generated subframe of its Henkin frame consists of a single MCS containing all nominals — the axiom schema n drives in all nominals immediately. Note that for any universal tense A , all instances of An are derivable in $K_{nt} + n$; simply apply the appropriate sequence of temporal generalisations. We then get the following transference: all instances of $A\phi \vee A\neg\phi$ are derivable, for all universal tenses A and wffs ϕ .

1. ϕ	Assumption	
2. $i \wedge \phi$	1; Axiom, PC	1
3. $i \wedge \phi \rightarrow A(i \rightarrow \phi)$	NOM _W variant	1
4. $A(i \rightarrow \phi)$	2, 3; MP	1
5. Ai	Theorem	1
6. $A\phi$	4, 5; TMP	1
7. $\phi \rightarrow A\phi$	1, 6; Discharge	
8. $\neg\phi \rightarrow A\neg\phi$	(Similarly)	
9. $\phi \vee \neg\phi$	PC	
10. $A\phi \vee A\neg\phi$	7, 8, 9; PC	

In particular, this means we can prove all instances of $Ap \vee \neg Ap$ for any variable p . This cannot be done in K_{nt} , thus there has been transference from the nominal schema to the variables. Two other simple examples are provided by the axiomatisations $K_{nt} + (n \wedge Fn)$ and $K_{nt} + (n \wedge \neg Fn)$. The first is complete with respect to the class of all singleton reflexive frames, the second with respect to the class of all singleton irreflexive frames — the proofs are straightforward. We first note that $K_{nt} + (n \wedge Fn)$ has as

theorems all instances of the T schema:

1. ϕ	Assumption	
2. $i \wedge Gi \wedge \phi \rightarrow G\phi$	Minimal theorem	1
3. i	Axiom	1
4. Gi	3, GGen	1
5. $G\phi$	1, 2, 3, 4 MP	1
6. $G(i \wedge \phi)$	4, 5 TL	1
7. Fi	Axiom	1
8. $F\phi$	6, 7 TMP	1
9. $\phi \rightarrow F\phi$	1, 8 Discharge	

(The minimal theorem in line 2 is a semi-contraposed instance of the end effect $i \wedge Gi \wedge F\phi \rightarrow \phi$ derived in Chapter 4.)

In contrast, $K_{nt} + (n \wedge \neg Fn)$ transfers all instances of the irreflexivity schema I to arbitrary wffs. That is, we can prove any instance of $\phi \rightarrow \neg F\phi$, including instances in variables such as $p \rightarrow \neg Fp$.

1. ϕ	Assumption	
2. $F\phi$	Assumption	1
3. Gi	Theorem	1, 2
4. $F(i \wedge \phi)$	2, 3; Inst	1, 2
5. Fi	4; TL	1, 2
6. $\neg Fi$	Axiom	
7. $\phi \rightarrow \neg F\phi$	1, 6; Discharge twice	

In passing, I would like to find a result determining exactly which logics can be so axiomatised. ²² I'll also add that there are many related problems that could broadly be classified as 'transference problems', and I find them some of the most fascinating open

²²I think such a result can be found, for I think transference is possible only under very limited circumstances. This is suggested both by the use of end effects to generate the above examples,

questions about NTL; however my present interest in transference problems is simply as an example of a question peculiar to sorted systems. Consider an arbitrary Δ -sorted language. Similar questions arise in this setting, but in general they will be more difficult. In the case of NTL we only had two sorts, and one of these was the unconstrained sort sort we call variables. In the more general case we will have multiple sorts, the subsort relation will be complex, and there may be no unconstrained sort. Investigating the question of transference will bring to the fore the difficulties involving substitutivity.

Matters get more difficult yet: some Δ -sorted languages are, intuitively, much better behaved than others. Consider any language of TREF. As far as completeness theory is concerned such languages are very well behaved indeed: the choice of operators and sorts seems a good one. On the other hand, consider a tensed language with 2-nominals. As we have seen, the completeness theory promises to be messy; and at a more intuitive level, the tense operators don't seem to be able to do much with 2-nominals. Adding the shifter doesn't seem to improve matters much — but adding the D operator does. For example

$$i_2 \wedge \phi \rightarrow D(i_2 \wedge \neg D(i_2 \wedge \neg \phi))$$

becomes valid: D can find the other end of the invisible rope linking the denotation of the 2-nominals.

Such considerations suggest to me the need to investigate alternative semantics. Firstly, for some Δ -sorted languages these investigations will undoubtedly be difficult — and the more analytic tools available the better. This is an uninteresting (though compelling) argument. However the real reason for the shift is motivated by the suspicion that there may really be a general theory of sorting. It seems unsatisfactory that one should have to build one's logical theory from scratch for each new sort — are there no general results which allow some interesting theory transfer? Further, is there any math-

and by the fact that to axiomatise the logics of the previous chapter, we only ever had to add to the standard tense logical axiomatisations some combination of $n \rightarrow \neg Fn$, $n \rightarrow G(Fn \rightarrow n)$, or $Pn \vee n \vee Fn$.

ematical content to the intuition that some sort/operator combinations are particularly natural? I don't know, but I believe the matter to be worth investigating — and I am dubious that an investigation in frame theoretic terms alone will be revealing. Accordingly, I'm going to outline two alternative semantic bases for our languages: quasi-frame based semantics, and algebraic semantics.

Quasi-frame semantics, for languages of NTL, is a minor variant of the frame based semantics we are used to: a *quasi-frame* is simply a frame with distinguished elements, and these elements are the only elements the nominals can denote. There are three reasons which suggest that quasi-frames are worth closer examination. The first is that they have a notion of validity preserving morphism; the second is that in addition to a global notion of validity, they possess a local notion of validity which captures some of the phenomena noted in Chapter 3 and 4; and thirdly they naturally give rise to operations which may yield specifically nominal preservation results.²³

Definition 6.3.1 (Quasi-frames and quasi-models) *Let \mathbf{T} be a frame and $I \subseteq T$ be non-empty. The pair $\langle \mathbf{T}, I \rangle$ is called a quasi-frame based on \mathbf{T} . If $\langle \mathbf{T}, I \rangle$ is a quasi-frame and $V \in \text{Val}(\mathbf{T})$ is such that $\bigcup_{i \in \text{NOM}} V(i) \subseteq I$, then $\mathbf{M}^I = \langle \langle \mathbf{T}, I \rangle, V \rangle$ is called a quasi-model on $\langle \mathbf{T}, I \rangle$. A quasi-frame is said to be finite if \mathbf{T} is finite, and quasi-finite if I is finite.* \square

We say that a wff ϕ is true at a point t in a quasi-model $\langle \langle \mathbf{T}, I \rangle, V \rangle$ iff $\langle \mathbf{T}, V \rangle \models \phi[t]$. We say ϕ is valid on a quasi-frame $\langle \mathbf{T}, I \rangle$ iff for all quasi-models $\langle \langle \mathbf{T}, I \rangle, V \rangle$ on $\langle \mathbf{T}, I \rangle$, and all $t \in T$, $\langle \langle \mathbf{T}, I \rangle, V \rangle \models \phi[t]$, and we write $\langle \mathbf{T}, I \rangle \models \phi$. It is clear that if ϕ is any wff, and $\text{card}\{i \in \text{NOM} : i \text{ occurs in } \phi\} = m$, then $\mathbf{T} \models \phi$ iff $\langle \mathbf{T}, I \rangle \models \phi$ for all \mathbf{T} based quasi-frames $\langle \mathbf{T}, I \rangle$ where I has cardinality at most $\max\{1, m\}$. This means that as far as the validity of single formulas is concerned, we need only look at quasi-finite quasi-frames.

²³The idea of isolating quasi-frames and quasi-models as semantic structures was suggested to me by Kit Fine. The observation that they have a notion of validity preserving morphism is also his.

Definition 6.3.2 Let $\langle S, I^S \rangle$ and $\langle T, I^T \rangle$ be quasi-frames. If $f : S \rightarrow T$ is a p -morphism from S to T such that $f^{-1}[I^T] \subseteq I^S$, and for all $i_s, i'_s \in I^S$, $i_s \neq i'_s$ and $f(i_s) = f(i'_s)$ implies that $f(i_s) \notin I^T$, then f is called quasi p -morphism (qp-morphism) from $\langle S, I^S \rangle$ to $\langle T, I^T \rangle$. \square

Given this definition and the discussion of p -morphisms in Chapter 2 we have that:

Lemma 6.3.1 If there is a qp-morphism f from $\langle S, I^S \rangle$ to $\langle T, I^T \rangle$, then $\langle S, I^S \rangle \models \phi$ implies $\langle T, I^T \rangle \models \phi$. \square

Intuitively, quasi-frames provide an inherently ‘local’ semantics for NTL: nominals can only denote points in I , hence (as nominals and variables have different expressive powers) we can talk about the I -regions of frames differently from the rest of the frame. Let’s make this precise.

If $\langle T, <, I \rangle$ is a quasi-frame then $\langle I, < \cap I^2 \rangle$ is a subframe of T . We say that ϕ is *quasi-valid* on $\langle T, <, I \rangle$ iff $\langle I, < \cap I^2 \rangle \models \phi$, and write $\langle T, <, I \rangle \models_q \phi$. Quasi-validity is local validity. For example, consider the quasi-frame

$$\langle Z \cup \{*\}, \{*\}, <_z \cup \{(\langle *, * \rangle)\},$$

where Z is the integers and $<_z$ their usual ordering. The wff $i \rightarrow Fi$ is not valid on this quasi-frame (all the elements of Z are irreflexive), but it is quasi-valid as $\{*\}$ is a region of local reflexivity. Quasi-validity gives rise to a notion of local definability: we say that ϕ quasi-defines a class of quasi-frames \mathcal{Q} iff for all quasi-frames $\langle T, I \rangle$, $\langle T, I \rangle \models_q \phi$ iff $\langle T, I \rangle \in \mathcal{Q}$. For example, $i \rightarrow \neg Fi$ quasi-defines the class of quasi-frames $\langle T, I \rangle$ such that I is irreflexive — we call these the quasi-irreflexive frames.

In various forms this sort of local validity has permeated our discussion of NTL. Consider the completeness result we proved for the logic PO in Chapter 4. We first generated a Henkin model, but this was not necessarily antisymmetric. What we did know, however, was that any point in this frame containing a nominal was a simple cluster. In the terms of the present discussion this amounts to saying that

$$\langle \langle H_{\Sigma}^{PO}, \{h : \exists i \in NOM, \text{ such that } i \in h\} \rangle, V \rangle,$$

(where \mathbf{H}_Σ^{PO} is the doubly generated Henkin frame and V the natural valuation), is a quasi-model. Moreover, the Simple Cluster Lemma tells us that it's a special quasi-model, for by that lemma we know that

$$I = \{h : \exists i \in NOM, \text{ such that } i \in h\}$$

is antisymmetric. That is, the quasi-frame is quasi-antisymmetric, and our model generation process has produced a quasi-frame which validates all the schemas in arbitrary wffs, and quasi-validates the purely nominal schemas. This type of phenomenon is perfectly general for all the completeness results we have considered.

We can also recast the decidability results of Chapter 4 in quasi-frame terminology. For example, the result for PO showed that this logic has the *finite quasi-frame property* with respect to the class of quasi-antisymmetric frames. In short, many of the constructions we have already met can be more naturally viewed as methods of making quasi-frames and quasi-models which quasi-validate certain formulas.

Finally, the notion of a quasi-frame naturally suggests certain operations; and it is worth investigating whether these give rise to interesting quasi-validity preservation results. Two give two examples, given a quasi-frame $\langle \mathbf{T}, I \rangle$ it is natural to ask whether there are interesting order theoretic constraints on how we may decompose I into two non-empty sets K and J in such a fashion that the quasi-validity of ϕ on $\langle \mathbf{T}, I \rangle$ is transferred to the quasi-frames $\langle \mathbf{T}, J \rangle$ and $\langle \mathbf{T}, K \rangle$. In a similar manner we can look for interesting necessary or sufficient conditions governing when merging two quasi-frames $\langle \mathbf{T}, I \rangle$ and $\langle \mathbf{T}, J \rangle$ to form $\langle \mathbf{T}, I \cup J \rangle$ is a quasi-validity preserving operation. I am trying to use these ideas to find a criterion for distinguishing those conditions definable in purely nominal NML, from those definable in purely nominal NTL.

It is straightforward to define quasi-frame analogs for other sorts, and these also provide 'local semantics' that may prove useful heuristics for investigating the properties of various sorts. Nonetheless, while I do expect quasi-frames to yield further insight into NTL, and expect that such a local semantics will always be a useful addition for investigating any particular sort, I doubt that such semantics will be of much help in formulating or solving more general problems: we have probably remained too close to

our original frame based semantics to obtain any really new insights into sorting. If we are to investigate more general issues I believe we should turn to algebraic semantics.

Historically algebraic semantics for standard tense and modal languages have proved invaluable, especially since the discovery of the frame incompleteness results. Duality theory, the study of the inter-relationships between algebraic constructions and the more familiar constructions on frames (or general frames), has provided answers to many difficult questions.²⁴ The success of this algebraic semantics for standard languages suggests it may be useful to develop sorted equivalents. I'll now sketch what is involved.

Let $\mathcal{B} = \langle B, 0, 1, -, +, \times, p, f \rangle$ be a *temporal algebra* as defined by S. K. Thomason [103]. That is, $\langle B, 0, 1, -, +, \times \rangle$ is a Boolean algebra — B is the carrier set, whose elements are represented by b, b', b'', \dots ; 0 is bottom; 1 top; and the operators have the obvious readings — and p and f are unary operators on B such that $f(0) = p(0) = 0$; $f(b + b') = f(b) + f(b')$; $p(b + b') = p(b) + p(b')$; and $f(b) \times b' = 0$ iff $b \times p(b') = 0$.

How can we adapt this semantics to languages with nominals? Let's first examine a plausible, but misguided, attempt. By an *atomic temporal algebra* is meant a temporal algebra whose underlying algebra is atomic; that is, for each $b \in B$ such that $b \neq 0$ there is an atom $i \in B$ such that $i \leq b$, where \leq is the usual operator induced ordering on the Boolean algebra.²⁵ As the use of i to denote atoms suggests, we are going to interpret nominals on these structures as atoms. Given any wff ϕ of NTL we form the corresponding *temporal polynomial* h_ϕ by replacing each propositional variable p_m by the corresponding variable p_m over arbitrary elements of our Boolean algebra; replacing each nominal i_i by the corresponding variable i_i over atoms; and replacing \neg, \vee, \wedge, F and P by $-, +, \times, f$ and p respectively. We say that an atomic Boolean algebra \mathcal{B} validates ϕ ($\mathcal{B} \models \phi$) iff $h_\phi = 1$ identically in \mathcal{B} .

²⁴An encyclopedic overview of duality theory, one which greatly extends and generalises the existing theory, may be found in Goldblatt's recent [41]. An interesting feature of this monograph is the way it concentrates on the dualities between algebras and frames, not general frames.

²⁵An atom in a Boolean algebra is a minimal non-zero element

That the NOM_W schema is sound in this interpretation follows by induction on the length of existential tenses,²⁶ but we don't get much further than this. Firstly note that it is not easy to generalise the kind of 'structural selection' of temporal algebras, exemplified in our isolation of the atomic modal algebras, to other sorts. Further, we have no easy way of building such algebras, save, as we shall later see, building them from frames. In particular, Lindenbaum algebras are not atomic temporal algebras.²⁷ We have stuck too closely to the intuitive interpretation of nominals, and the algebraic structure is not doing any work for us. Let us liberalise our class of algebras by letting the algebraicisation of the NOM_W schema select the particular temporal algebras required.

Accordingly we define a *nominal temporal algebra* to be a pair $\mathcal{N} = \langle \mathcal{B}, I \rangle$ where \mathcal{B} is a temporal algebra and I a non-empty subset of B such that for all $i \in I$, for all $b \in B$ and for all compositions e of the f and p operators (including the null composition), $i \times e(i \times b) \times -b = 0$. Given such an algebra we interpret a language of NTL on it in the obvious fashion. That is, we form our polynomials as before, and insist that the variables i_i in these polynomials that correspond to nominals range over all and only the elements

²⁶In the inductive proof that follows we write the algebraic counterpart of an existential tense E by e — an m length sequence of existential tenses corresponds to the m composition of the corresponding algebraic operators. The base case is to show that $i \times (i \times b) \times -b = 0$, which is clear. Suppose that for all e such that $\text{len}(e) < n$, $i \times e(i \times b) \times -b = 0$. Now suppose that E' is an existential tense of length n . We want to show that $i \times oe(i \times b) \times -b = 0$, where o is either p or f . As i is an atom in the algebra, for any element $b \in B$, $i \times b = i$ or $i \times b = 0$, so if $i \times oe(i \times b) \times -b \neq 0$, it must equal i instead. But assuming it to equal i leads to a contradiction as follows. Firstly, this immediately implies that $i \times -b = i$. But the inductive hypothesis is that $i \times e(i \times b) \times -b = 0$, and substituting $i \times -b$ for i means that $i \times oe(i \times -b \times b) = i$, but this means $oe(0) = i$, which is impossible.

²⁷The problem, using the standard notation introduced below, is that for all $i \in \text{NOM}$, $[i]$ is not an atom as (for example) $[i \wedge j] < [i]$, $[i \wedge j] \neq [i]$, and $[i \wedge j] \neq [\perp]$, where j is any nominal distinct from i .

of I . In short, we have moved to a two sorted algebraic semantics — the sorts in question being B and I — and when evaluating our polynomials we only consider the sortally correct evaluation possibilities. As before, we say that a nominal temporal algebra \mathcal{B} validates ϕ ($\mathcal{B} \models \phi$) iff $h_\phi = 1$ identically in \mathcal{B} ; and we say that ϕ is algebraically valid iff $\mathcal{B} \models \phi$ for all nominal temporal algebras \mathcal{B} . Note that all atomic temporal algebras are nominal temporal algebras.

Given any algebra $\mathcal{N} = \langle \mathcal{B}, I \rangle$ of the same type as a nominals temporal algebra — that is, \mathcal{B} is an algebra of the same type as a temporal algebra, and I is a non-empty subset of \mathcal{B} 's carrier set B — we can interpret a wff ϕ on \mathcal{N} exactly as though \mathcal{N} were a nominal temporal algebra. However, following Thomason, we have that the nominal temporal algebras are precisely the algebras that validate the K_{nt} theorems:

Lemma 6.3.2 *Let $\mathcal{N} = \langle \mathcal{B}, I \rangle$ be an algebra of the type of nominal temporal algebras. Then \mathcal{N} validates all theorems of K_{nt} iff \mathcal{N} is a nominal temporal algebra.*

Proof:

Suppose $\langle \mathcal{B}, I \rangle$ is a model for K_{nt} . By Thomason's proof \mathcal{B} must be a temporal algebra. Only the part peculiar to the nominals thus remains, and this is immediate: if $\langle \mathcal{B}, I \rangle$ is a model for K_{nt} it validates all instances of $i \wedge E(i \wedge \phi) \rightarrow \phi$, that is,

$$\neg(i \wedge E(i \wedge \phi) \rightarrow \phi) \leftrightarrow \perp,$$

which means that in $\langle \mathcal{B}, I \rangle$, $i \times e(i \times b) \times \neg b = 0$, for all $i \in I$, $b \in B$, and composition sequences e .

Proving that any nominal temporal algebra is a model for K_{nt} merely involves the usual inductive soundness argument. \square

This shows that the set of all algebraically valid NTL wffs is identical to the set of wffs valid on all frames.

We can build nominal temporal algebras easily, and without any reference to the frame based semantics, by using the Lindenbaum construction. There is a small catch:

examination of (say) Lemmon's proof [60, page 51] that this can be done for standard modal languages reveals that the proof depends on the fact that standard intensional logics are closed under uniform substitution, something that is not the case in NTL. However we can impose a weak (and natural) substitutivity closure condition on logics of NTL, which, because our algebras are two sorted, enables the construction to perform its task. In fact, if we take this closure condition as part of what we mean by a nominal tense logic, we can prove a Thomason style adequacy theorem for the algebraic semantics.

Recall from Chapter 5 that a wff ψ is said to be obtained from a wff ϕ by NTL substitution iff ψ is the result of uniformly substituting arbitrary NTL wffs θ for variables in ϕ , and uniformly substituting nominals for nominals in ϕ . We define a *nominal tense logic* L to be a set of wffs such that all K_{nt} theorems are in L , and L is closed under modus ponens, temporal generalisation, and NTL substitution. This is not unduly restrictive: there are many interesting nominal tense logics, and in particular, the theory of any class of frames is a nominal tense logic — the proof of this is left to the reader. This enables us to prove:

Theorem 6.3.1 (Adequacy Theorem) *For any nominal tense logic L there is nominal tense algebra $\mathcal{N}^L = \langle \mathcal{B}, I \rangle$ such that $\mathcal{N}^L \models \phi$ iff $\vdash_L \phi$.*

Proof:

let \mathcal{N} be the Lindenbaum algebra for L . That is, $B = \{[\phi] : \phi \in WFF\}$; $[\phi] = [\psi]$ iff $\vdash_L \phi \leftrightarrow \psi$; $0 = [\perp]$; $1 = [\top]$; $-[\phi] = [\neg\phi]$; $[\phi] + [\psi] = [\phi \vee \psi]$; $[\phi] \times [\psi] = [\phi \wedge \psi]$; $f([\phi]) = [F\phi]$; $p([\phi]) = [P\phi]$; and $I = \{[i] : i \in NOM\}$. By the usual reasoning $\mathcal{B} = \langle B, 0, 1, -, +, \times, f, p \rangle$ must be a temporal algebra. Moreover, as K_{nt} includes among its theorems all instances of NOM_W and \top , we have that any instance of NOM_W is equivalent to \top in K_{nt} , and hence in L . This means that for all $[i] \in I$, $[\phi] \in B$ and composition sequences e we have in the Lindenbaum algebra that $-([i] \times e([i] \times [\phi])) + [\phi] = 1$, or $[i] \times e([i] \times [\phi]) \times -[\phi] = 0$. Thus the Lindenbaum algebra is a nominal temporal algebra and hence validates all theorems of K_{nt} . The key point, however, is to show that the

Lindenbaum algebra validates not just the K_{nt} theorems, but all L theorems, and here is where we use the fact that nominal tense logics are closed under NTL substitution. Let ϕ be any wff. When we evaluate the polynomial h_ϕ on the Lindenbaum algebra, each evaluation yields a value $[S(\phi)]$, where $S(\phi)$ is an NTL substitution instance of ϕ . (This follows by induction on $\deg(\phi)$.) So suppose $\vdash_L \phi$. As L is closed under NTL substitutions, $\vdash_L S(\phi)$ for all NTL substitution instances $S(\phi)$ of ϕ . Hence for all such $S(\phi)$, $[S(\phi)] = 1$, thus $h_\phi = 1$ identically, and L 's Lindenbaum algebra validates every L theorem.

To show that any non-theorem ϕ of L is falsified in this algebra, suppose that $\not\vdash_L \phi$. If for each variable p_m and nominal i_i in ϕ we evaluate h_ϕ with the corresponding polynomial variables interpreted by $[p_m]$ and $[i_i]$ respectively, we obtain $[\phi]$. Clearly $[\phi] \neq [1]$ as otherwise $\vdash_L \phi$, which contradicts our assumption of ϕ 's non-theoremhood. \square

We now must link our frame based semantics with our algebraic semantics; the work involved is standard. Firstly, each frame $\langle T, < \rangle$ gives rise to a temporal algebra

$$\mathcal{B} = \langle Pow(T), \emptyset, T, \setminus, \cup, \cap, \pi_p, \pi_f \rangle,$$

where \setminus, \cup and \cap are the usual set theoretic operations, and π_p and π_f are as defined in Chapter 2. In order to make a nominal temporal algebra out of this we merely form the pair $\mathcal{N} = \langle \mathcal{B}, I \rangle$, where I is the set consisting of all and only the singleton subsets of T . Note that this algebra is an atomic temporal algebra.

As for the other direction, every algebra gives rise to a frame via Stone representation. Given a nominal temporal algebra $\langle \mathcal{B}, I \rangle$, we define $F(\langle \mathcal{B}, I \rangle)$ to be $\langle Ultra(\mathcal{B}), <_U \rangle$, where $Ultra(\mathcal{B})$ is the set of all ultrafilters U on \mathcal{B} , and $<_U$ is defined by $U_1 <_U U_2$ iff for all $b \in \mathcal{B}$, $b \in U_2$ implies $f(b) \in U_1$. It is not difficult to show that in any frame $F(\langle \mathcal{B}, I \rangle)$ manufactured by this process, no distinct ultrafilters in $Ultra(\mathcal{B})$ containing common elements of I can be in the same generated subframe; the proof is essentially an algebraic reformulation of the proof of the Unique Occurrence Lemma. This last observation opens up the topic of how we make models, and general frames out of algebras. I am not going to discuss this, but will content myself by pointing out the following (unsurprising)

relation between the Henkin frame of an arbitrary nominal tense logic L and the Stone representation of L 's Lindenbaum algebra: they're isomorphic. As the reader knows how to build models out of Henkin frames, this result should make clear the essence of what is involved in building models from algebras.

Theorem 6.3.2 *Let L be a nominal tense logic. Let $F(\mathcal{N}^L)$ be the frame constructed out of the Lindenbaum Algebra for L as defined above, and let \mathbf{H}^L be the canonical Henkin frame for L . Then $F(\mathcal{N}^L) \cong \mathbf{H}^L$.*

Proof:

Define $g : H^L \rightarrow F(\mathcal{N}^L)$ by $g(h) = \{[\phi] : \phi \in h\}$, for all $h \in H^L$. This is a well defined function: for all $h \in H^L$, $g(h)$ is a subset of the carrier set of the Lindenbaum Algebra of L ; and because of the familiar closure properties on MCSs, it is always an ultrafilter.

That g is injective is clear. To see that it is also surjective, for all $U \in F(\mathcal{N}^L)$ define MAX_U to be $\bigcup\{u : u \in U\}$. We claim MAX_U is an MCS and that $g(MAX_U) = U$. That for all $U \in F(\mathcal{N}^L)$, MAX_U is set of wffs is clear. It cannot be an inconsistent set, for this would mean that for some wff ϕ both ϕ and $\neg\phi$ were in MAX_U . This would mean that for some $u, u' \in U$, $\phi \in u$ and $\neg\phi \in u'$, which in turn would mean that $[\phi] = u$ and $-\phi = u'$, but both of these cannot belong to any ultrafilter. Similarly, MAX_U must be maximal, for as U is an ultrafilter either $[\phi]$ or $-\phi \in U$, hence ϕ or $\neg\phi \in MAX_U$. Thus MAX_U is always an MCS. To see that for all $g(MAX_U) = U$ we need merely observe that it is clear from the definition of g that $g(MAX_U) \subseteq U$. But as both $g(MAX_U)$ and U are ultrafilters it must be that $g(MAX_U) = U$. In short, g is surjective, and MAX is the inverse g^{-1} of g .

Moreover g is a frame isomorphism. Firstly, for all $h, h' \in H^L$, $h <_h h'$ iff $\phi \in h'$ implies $F\phi \in h$. But then it is immediate from the definition of g that if $h <_h h'$ then $g(h) <_U g(h')$, where $<_U$ is the ordering on $F(\mathcal{N}^L)$. Conversely suppose that $g(h) <_U g(h')$, and that $\phi \in h'$. Then $[\phi] \in g(h)$, hence by the definition of $<_U$, $f[\phi] \in g(h')$. But $g^{-1}(g(h')) = MAX_{g(h')}$, hence $F\phi \in MAX_{g(h')}$, and so $h <_h h'$, which is what we wanted to show. Thus g is an isomorphism. \square

This concludes our discussion of nominal temporal algebras. Using it as a guide the reader will find it straightforward to define the notion of an interval nominal algebra, and prove analogous results concerning them. Let's conclude the chapter by defining an algebraic semantics for Δ -sorted languages.

Let Δ be a non-empty set. By a Δ -sorted temporal algebra is meant a pair $\langle \mathcal{B}, \{I_\delta : \delta \in \Delta\} \rangle$, where \mathcal{B} is a temporal algebra, and for each $\delta \in \Delta$, I_δ is a non-empty subset of B , the carrier set of \mathcal{B} . We denote such an algebra by \mathcal{B}^Δ . Given a Δ -sorted language \mathcal{L}^Δ , we evaluate wffs ϕ of \mathcal{L}^Δ on a Δ -sorted algebra \mathcal{B}^Δ by evaluating the sorted polynomial h_ϕ in the expected way: variables in h_ϕ of sort S_δ — that is, the variables in the polynomial which correspond to atoms of sort S_δ — are taken as ranging over all and only the elements of I_δ . We say $\mathcal{B}^\Delta \models_\Delta \phi$ iff $h_\phi = 1$ identically in \mathcal{B}^Δ ; and given any class of Δ -sorted algebras C_Δ , we say that $C_\Delta \models_\Delta \phi$ iff for all $\mathcal{B}^\Delta \in C_\Delta$, $\mathcal{B}^\Delta \models_\Delta \phi$.

This semantics is simpler^{than} the apparatus of Δ -sorted frames and interpretational bases. Moreover, sorts are algebraically reflected by multisorted equations over the algebras; this immediately suggests a number of questions. Which standard operations on algebras preserve which equations, and given an algebra satisfying a set of equations involving sorts in Δ , when can a set of equations involving sorts $\Delta \subset \Delta'$ pick out the same algebra? (This seems to be the algebraic analog of transfer.) Whether the algebraic approach will help us to gain interesting answers to the more general questions posed in this chapter I don't know; I think it's a sensible place to start looking.

This brings the chapter to an end, and with it a major portion of the thesis: with the exception of a brief excursion into sorted interval based logics in the final chapter, our logical work is done. I believe this work has shown that sorting is of logical interest, and strongly suggests that much work of interest remains. But now it is time to turn to a new topic: the applicability of sorting.

Chapter 7

Applications in Natural Language Semantics

In this chapter referential sorting is used to model aspects of tense, temporal reference, and their interaction. In the first section we sketch the referential, or Reichenbachian, account of natural language tense, and show that TREF accommodates it rather well. In the second section a series of four languages, \mathcal{L}_c^0 through \mathcal{L}_c^3 , is presented. Each is an extension of TREF, and increases its coverage in a uniform way: these languages have new atoms — such as *now*, 1999, *Monday* and *yesterday* — which may be freely combined in the usual manner. These extensions cope straightforwardly with a number of puzzles such as the mismatch between tense and indexical reference in “John will run yesterday”.

In the course of this discussion, matters of more general concern emerge, and these are discussed in the third section. When the atoms *now*, *today*, *tomorrow* and *yesterday* were added to TREF in the second section, heavy use was made of one of the central ideas of the California theory of reference [68][69][52] [56][55], contextual evaluation. Some technical changes were introduced — the contextual apparatus was exploited by means of sorting, not new operators, for one thing — but these additions were intended to be in the spirit of the Californian tradition, and I believe they are. Now, work in a field that has come to be known as two dimensional logic also took as its starting point the Californian idea of evaluating expressions at a pair of indices. While this tradition is technically interesting, it has had two unfortunate effects. Firstly it has obscured

the semantic insights afforded by the California theory of reference — indeed obscured them to the point where these insights are seen as merely amounting to the discovery of primitive forms of two dimensional logic. Secondly the complexity of these logics, and their complete lack of natural language motivation, has made it appear that tense logic has little to offer natural language semantics. In this section I show that plain (‘one dimensional’) TREF handles the problems that motivated such extensions as multiple Vlach operators.

The fourth section discusses the application of referential sorting to discourse phenomena. It is noted that nominals and interval nominals can be used as discourse markers to model the some of types of phenomena discussed by Partee [75], such as indefinite antecedent anaphora, bound variable uses, and ‘temporal donkeys’. The relationship between TREF and temporal DRT is briefly discussed.

7.1 Tense and Reichenbach

Standard Priorean tense logic contains no mechanism for referring to times; it can merely quantify over them in restricted fashion using its operators. This means that its use as a model of tense in natural language has severe limitations, even when it comes to modeling such apparently unproblematic tenses as the simple past and simple future of English; these tenses aren’t solely, or even primarily, quantificational in nature. Barbara Partee [75, pages 244–245], for example, compares the use of the past tense in the sentence “I didn’t turn off the stove” (uttered without previous linguistic context while driving down a freeway) to the way third person pronouns function: both devices refer to some entity that is salient to the listener. In this sentence the salient entity is a time: possibly a point, more likely an interval, and perhaps occurring just before the drive down the freeway began. This means that the usual purely Priorean representation of this sentence,

$$P(\neg \text{I turn off the stove}),$$

is not particularly good, for the act of temporal reference is uncaptured. Far too many points of time are relevant to the truth value of this wff. (Presumably there is no stove

in the car, so no act of stove-turning-off can have occurred during the journey — but this means that any time during the duration of the journey makes the above wff true in a way that is irrelevant to the meaning of the original sentence.) In contrast, in a language of TREF we could write down

$$P(i \wedge \neg \text{I turn off the stove});$$

we have used a nominal to refer to a specific time, asserted of that time that no stove-turning-off took place then, and have thus captured the required element of referentiality. Since the time of ‘not turning off the stove’ is probably associated with the rather fuzzy interval centred round the preparations for the journey, the use of an interval nominal, yielding

$$P(e \wedge \neg \text{I turn off the stove}),$$

may be preferred.

Let’s consider another example. In the same paper Partee considers the sentence “When John saw Mary, she crossed the street” which she considers to be analogous to the paradigmatic case of pronominal anaphora;¹ the first clause picks out a time, and the second anchors the crossing of the street to this time. Representing this inter-clausal shared temporal reference is crucial, and is easily accomplished in TREF:

$$P(i \wedge \text{John sees Mary}) \wedge P(i \wedge \text{Mary crosses the street}).$$

Actually this wff is probably better regarded as an encoding of the two sentence discourse “John saw Mary. She crossed the street.” The single clause variant given above might be better represented as

$$P(i \wedge \text{John sees Mary} \wedge \text{Mary crosses the street}).$$

¹That is, where an antecedent noun phrase picks out a particular individual and a later pronoun is used to denote the same individual. For example, “Gordon is feeling unwell. He has the ‘flu.’”

Of course these two wffs are logically equivalent — note that the implication from the first to the second is an instance of NOM — but the function of the ‘When’ here seems precisely to perform a NOM style ‘compression’ of the matrix of the second clause into the tense structure created by the first, and presumably this is something a good translation function would do.

These referring uses of simply tensed sentences are not isolated. It would be an exaggeration to call them canonical, but in general it is easier to construct simple sentences in which the tense does refer rather than ones in which it doesn’t.² Some sentences don’t make full use of this referential potential, but tenses can and usually do refer.

Such observations suggest the following. In general the tense of a simply tensed sentence seems to embody at least two types of information. Firstly it encodes a ‘shift’ — though a *directed* shift, unlike the shift of L — an instruction to move (or search) forwards or backwards in time according as the tense is future or past.³ Secondly it encodes a (perhaps not particularly tight) specification of which temporal region in the indicated direction the event under consideration took place at. This suggests the following heuristic: we should try to break tensed information apart into a ‘shift’ and a ‘reference’ and code the result in TREF. For example, simple tensed sentences should be encoded in the form $P(r \wedge \phi)$ or $F(r \wedge \phi)$, where ϕ is the ‘event matrix’ — the condition asserted to hold — and r is one of our TREF referring atoms. (Usually r will be a nominal or an interval nominal; but in order to cope with any genuine cases where simply tensed sentences have no referential import we allow \top to count as a referring

²As an example of a sentence with a tense that doesn’t seem to refer, consider “All persons alive now will be dead”. Other possible candidates for non-referential use of tense are “Modus ponens will be valid”, and “One plus one was two”. Another source arises from certain uses of the simple future, exemplified by “If something isn’t done about that tooth, you’ll have a bad time of it”. The use of the future tensed ‘you’ll’ doesn’t refer to a future time in any simple fashion; as is so often the case, the use of the future tense here is not purely temporal. Such uses raise interesting problems, but these will not be considered here.

³That is, “Look right!”, or “Look left!”.

atom.) If something along the lines of this heuristic is sustainable it would be very pleasant, for shifting and referring is what languages of TREF were designed to do. Does this construal of natural language tense lead anywhere interesting? Firstly, it links neatly with Reichenbach's account of tense [86, pages 287–298].

Reichenbach insisted that understanding tensed expressions involved understanding the temporal relations that can hold between *three* special points of time. The first two points, *point of speech* and *point of event* have the obvious meanings: for example, given an utterance of “John ran” at time t_0 , t_0 is the point of speech, and the time when John actually ran is the point of event.⁴

It is the third type of point, point of reference, that is Reichenbach's novelty. Consider the past perfect sentence “John had run”. Note the way it works — our attention is not directed immediately to the time at which John runs; rather we are first referred to some point t' preceding the point of speech and told that at some point prior to this intermediate point John ran. This intermediate point is called the point of reference; it is the ‘vantage point’ from which the event in the sentence is surveyed. Reichenbach accounts for the variety of tenses found in natural language in terms of the different patterns of temporal precedence and coincidence these three points can exhibit. For example, in the past perfect we have that $E < R < S$, or in the ‘diagram notation’ that Reichenbach uses, $E - R - S$.⁵ The simple past is characterised by Reichenbach by the pattern $E, R - S$; just as we have seen, in the simple past we refer to the time the event described took place.

⁴In general Reichenbach treats point of event as being a point of time rather than an interval. At one stage, [86, page 290] he does consider extended or intervallic events, but this possibility is only raised for sentences in the progressive. Allowing ‘point of event’ to be an interval seems a natural augmentation of his position however. A rather more interesting question is whether ‘point of reference’ (discussed next) can be an interval. In what follows I simply assume that this is the sensible thing to do. (Much later we'll see an example which indicates why.)

⁵Reichenbach indicates that a point P precedes a point P' by writing $P - P'$. If these two points coincide he writes P, P' .

Let's systematically consider the thirteen original Reichenbachian possibilities. The first three columns of the following table are due to Bernard Comrie [19, page 25]; they tabulate the possibilities admitted by Reichenbach's idea. The fourth column gives our representation of these tenses in TREF.

Structure	Name	English example	Representation
E-R-S	Pluperfect	I had seen	$P(r \wedge P\phi)$
E,R-S	Past	I saw	$P(r \wedge \phi)$
R-E-S	Future-in-the-past	I would see	$P(r \wedge F\phi)$
R-S,E	Future-in-the-past	I would see	$P(r \wedge F\phi)$
R-S-E	Future-in-the-past	I would see	$P(r \wedge F\phi)$
E-S,R	Perfect	I have seen	$P\phi$
S,R,E	Present	I see	ϕ
S,R-E	Prospective	I am going to see	$F\phi$
S-E-R	Future perfect	I will have seen	$F(r \wedge P\phi)$
S,E-R	Future perfect	I will have seen	$F(r \wedge P\phi)$
E-S-R	Future perfect	I will have seen	$F(r \wedge P\phi)$
S-R,E	Future	I will see	$F(r \wedge \phi)$
S-R-E	Future-in-the-future	(Latin: <i>abiturus ero</i>)	$F(r \wedge F\phi)$

Note that the perfect and past tenses are represented differently. The identical representations these two tenses receive in standard tense logic is one of the major sources of dissatisfaction with the use of tense logic in the analysis of natural language. That a distinction exists in TREF is pleasant, but it would be premature to attach a great deal of importance to this. The TREF representation has added something — a Reichenbachian component — to Prior's account of tense; but we have added nothing to the (very unsatisfactory) Priorean account of the perfect. I'll make a brief remark concerning the perfect at the end of the chapter.

Also note that all three logically possible permutations of *R*, *E* and *S* for each of the Future-in-the-past and the future perfect are represented by a single formula. Comrie

declares that such representations are essential [19, pages 26 – 27]: while one form may be favoured, and one largely ruled out, these are Gricean implicatures, all three usages are possible, and a uniform representation is needed.

The table does not exhaust all the possibilities, even for the case of English. Consider the example, due to Prior [79, page 13], “I shall have been going to see John”. This seems to require two points of reference, $R1$ and $R2$. In Reichenbach’s diagrams the most pragmatically likely pattern they will exhibit is $S - R2 - E - R1$. The semantic possibilities are captured in TREF by:

$$F(r \wedge P(r' \wedge F(\text{See John}))).$$

More generally, Comrie [19], [20, pages 122 – 130] proposes a modification of Reichenbach’s system which essentially amounts to the following: Reichenbach’s simultaneity relation is replaced by an intervallic overlap relation, there may be an arbitrary number of reference points, and the point of speech is linked to the point of event by a (possibly null) zig-zag sequence of reference points. This isn’t made completely precise, but I believe the following captures everything Comrie intended. We take as primitive tenses the three forms $P\phi$, ϕ , and $F\phi$. We then say that if ψ is a tensed form, so are $P(r \wedge \psi)$, $(r \wedge \psi)$, and $F(r \wedge \psi)$, for all referential atoms r , and inductively close the set in the usual fashion. The set of tensed forms seems to capture Comrie’s intention. (In fact it may overgenerate — it’s not clear to me whether Comrie wants closure under the form $(r \wedge \psi)$ — but this stipulation can be dropped.)

7.2 Indexicals and dates

While the use of nominals and interval nominals to model the deictic and anaphoric aspects of tense is a natural one, and while the link between the work of Prior and Reichenbach their use suggests is pleasant, a charge of triviality looms. Standard Priorean logic’s lack of referential mechanisms — so the charge runs — is a major defect. That languages of TREF with their two referential sorts can achieve more is a truism: given

such apparatus how could one fail to do better? Such success does not establish that the approach is an interesting one, and until this is shown the previous remarks remain isolated observations and the link with Reichenbach's work a piece of interpretational whimsy.

To rebut such a charge we must show that referential sorting can naturally model further phenomena and this is the principal task of the remainder of the chapter. In the present section we model two common types of temporal referring terms, indexicals and dates. I believe the results show that referential sorting is an interesting approach, and for two reasons.

The first is the simple and uniform mode of representation. All such terms — 'now', 'October', 'Monday', 'yesterday' and '1984' — are handled in syntactically uniform fashion: in particular, they are treated as atoms and freely combined with all the logical connectives just as nominals and interval nominals are. This will permit the sentence "John climbed yesterday" to be represented as

$$P(yesterday \wedge (\text{John climbs}));$$

and "John reached the summit of Ruapehu on Wednesday the 15th November, 1989" to be represented by

$$P(Wednesday \wedge 15 \wedge November \wedge 1989 \wedge AD \wedge (\text{John} \dots \text{Ruapehu}));$$

and some well known difficulties are cleared up on the way.

But there is another, deeper, reason for satisfaction: the use the semantic machinery of the California theory of reference is put to. In this section I will show that not only can we successfully sort in contextualised semantics, but that sorting is an elegant way to exploit double indexing.

We proceed as follows. Four contextualised languages are introduced, \mathcal{L}_c^0 to \mathcal{L}_c^3 . \mathcal{L}_c^0 is the base language; in \mathcal{L}_c^1 we add a name for 'now'; in \mathcal{L}_c^2 we add names for 'yesterday', 'today' and 'tomorrow'; and in \mathcal{L}_c^3 we add the paraphernalia of calendars.

7.2.1 The language \mathcal{L}_c^0

By the contextualised language of temporal reference \mathcal{L}_c^0 is simply meant the fixed choice of some language of TREF. That is, syntactically \mathcal{L}_c^0 has a denumerably infinite supply of variables $VAR = \{p, q, r, \dots\}$; a denumerably infinite supply of nominals, $NOM = \{i, j, k, \dots\}$; and a similar supply of interval nominals, $INOM = \{e, d, c, \dots\}$. As usual VAR , NOM and $INOM$ are pairwise disjoint, and $VAR \cup NOM \cup INOM$ is the set of \mathcal{L}_c^0 atoms. The wffs of \mathcal{L}_c^0 are inductively produced by boolean combination and application of the operators F , P and L . The respective dual operators G , H and M are defined as usual.

We give this language a slightly different semantics however. The semantics of \mathcal{L}_c^0 — and indeed of all the contextualised languages considered in this section — is ultimately in terms of *contextualised frames*. We augment our old temporal structures (frames) with a new set of indices called *contexts of utterance*, and link these new items with frames in the simplest possible way, namely by stipulating that each context of utterance occurs at some point in time.

Definition 7.2.1 (Contextualised frames) *By a contextualised frame (c-frame) \mathbf{T}^C is meant a 4-tuple $\langle T, <, C, g \rangle$ where:*

1. $\langle T, < \rangle$ is a frame;
2. $C \neq \emptyset$, and $T \cap C = \emptyset$;
3. $g : C \rightarrow T$.

C is called the set of contexts of utterance or contexts on \mathbf{T} . We often write contextualised frames as 3-tuples $\langle \mathbf{T}, C, g \rangle$, where \mathbf{T} is the frame $\langle T, < \rangle$, and we say that a c-frame \mathbf{T}^C is based on the frame \mathbf{T} . If \mathbf{T}^C and $\mathbf{T}^{C'}$ are c-frames based on the same frame \mathbf{T} then they are called contextual variants. \square

Thus we have our usual model of temporal flow $\langle T, < \rangle$, a collection C of contexts of utterance, and we know (via g) where each context is temporally situated. ⁶

⁶Note that although we have the shifter L in contextualised languages we have not explicitly

The simplifications introduced by this modeling hardly need elaborating: to the usual objections that a point based temporal ontology is inadequate we can add the complaint that introducing a simple set C as a model of the notion of context hardly counts as a profound analysis — at the very least we would expect contexts to ‘take time’, and thus perhaps g should assign intervals to elements of C . In fact we would demand an awful lot more of anything that was to count as an adequate model of context: we would certainly expect any such modeling to give contexts a rich internal structure, possibly including such features as agent(s), patient(s), and location(s); and we would want this internal structure to aid in imposing useful structure on the set of contexts — such useful (external) structure perhaps including relations such as sub-context, contextual overlap and causal linkage. In short, an adequate modeling of the notion of context would have a lot in common with the sort of structures demanded by situation theory [3]. Although in the course of this section we progressively remove some of the limitations of our modeling, we never attempt the difficult task of giving an adequate account of context. The main reason for this is that in order to model some of the more obvious features of natural language temporal reference and its interaction with tense, such modeling is unnecessary. For the simple matters considered here, all we need to know is that there are such things as contexts, and that they are situated in time.

We now link \mathcal{L}_c^0 with our new semantic structures:

Definition 7.2.2 *By a valuation for \mathcal{L}_c^0 on a c -frame $\mathbf{T}^C = \langle \mathbf{T}, C, g \rangle$ is meant a function*

$$V : \text{ATOM}_{\mathcal{L}_c^0} \times C \longrightarrow \text{Pow}(T),$$

included the universal relation; instead we will state L ’s truth condition directly. (The only reason we used the multiframe definition in the previous chapter was for its convenience when proving completeness results.) In passing, our gloss of C as ‘contexts of utterance’ shouldn’t be taken too literally; ‘contexts in which a sentence *might* be uttered’ is more accurate. Nonetheless, as long as we keep the (fairly obvious) limitations of our modeling in mind the gloss is both harmless and appealing.

such that for all $i \in NOM$, $V(i, c)$ is a singleton subset of T ; and for all $e \in INOM$, $V(e, c) \in Int(\mathbf{T})$. By an \mathcal{L}_c^0 model \mathbf{M}^0 is meant a pair $\langle \mathbf{T}^C, V \rangle$ where \mathbf{T}^C is a contextual frame, and V a valuation for \mathcal{L}_c^0 on \mathbf{T}^C . \square

Nominals and interval nominals no longer denote uniquely — however within any particular context of utterance (‘on any particular occasion of use’), they do. Note that the definition of valuation has a pleasant flavour of ‘local truth’: valuations V can assign different truth values to the same atom in two temporally coincident contexts. That is, both p and $\neg p$ can hold at the same time — as long as the contexts are different. It can be raining in Edinburgh, yet sunny in Glasgow.

Given an \mathcal{L}_c^0 model $\mathbf{M}^0 = \langle \mathbf{T}^C, V \rangle$ and an \mathcal{L}_c^0 wff ϕ , we define the notion of truth of ϕ in \mathbf{M}^0 at a point $t \in T$ relative to a context $c \in C$ as follows:

$$\begin{aligned} \mathbf{M}^0 \models a[t, c] & \quad \text{iff} \quad t \in V(a, c), \text{ for all atoms } a \\ \mathbf{M}^0 \models \neg\phi[t, c] & \quad \text{iff} \quad \text{not } \mathbf{M}^0 \models \phi[t, c] \\ \mathbf{M}^0 \models \phi \wedge \psi[t, c] & \quad \text{iff} \quad \mathbf{M}^0 \models \phi[t, c] \text{ and } \mathbf{M}^0 \models \psi[t, c] \\ \mathbf{M}^0 \models F\phi[t, c] & \quad \text{iff} \quad \exists t'(t < t' \text{ and } \mathbf{M}^0 \models \phi[t', c]) \\ \mathbf{M}^0 \models P\phi[t, c] & \quad \text{iff} \quad \exists t'(t' < t \text{ and } \mathbf{M}^0 \models \phi[t', c]) \\ \mathbf{M}^0 \models L\phi[t, c] & \quad \text{iff} \quad \mathbf{M}^0 \models \phi[t', c], \text{ for all } t' \in T. \end{aligned}$$

(We often talk of evaluating a wff ϕ at a pair $[t, c]$.) Note that none of the above clauses changes the value of c during the process of evaluating the truth value of wffs. That is, if we evaluate a wff ϕ in a context c at time t , the truth value of ϕ depends on the truth value of its subformulas at other context/time pairs, but *only* on pairs of the form $[t', c]$, where c is the context ϕ was originally uttered in. None of the clauses exploits the presence of the c coordinate in any fashion: contexts for \mathcal{L}_c^0 are idle additions. The following lemma, immediate by induction on $deg(\phi)$, makes this clear:

Lemma 7.2.1 *Let $\mathbf{T}^C = \langle \mathbf{T}, C, g \rangle$ be a c -frame, $\langle \mathbf{T}^C, V^C \rangle$ be an \mathcal{L}_c^0 model, and c some fixed element of C . Then if V is a TREF valuation on \mathbf{T} such that $V(a) = V^C(a, c)$, for all atoms a then:*

$$\langle \mathbf{T}^C, V^C \rangle \models \phi[t, c] \text{ iff } \langle \mathbf{T}, V \rangle \models \phi[t]$$

for all wffs ϕ . □

What is meant by validity for contextualised languages? There are (at least) three concepts with claim to the title.

Definition 7.2.3 (Validity in contextualised languages) *Let \mathbf{T}^C be the c-frame $\langle T, <, C, g \rangle$ and ϕ a wff of a contextualised language. Then ϕ is valid on \mathbf{T}^C iff for all valuations V on \mathbf{T}^C , for all $t \in T$ and all $c \in C$*

$$\langle \mathbf{T}^C, V \rangle \models \phi[t, c].$$

Further, ϕ is locally utterance valid on \mathbf{T}^C iff for all valuations V on \mathbf{T}^C , and all $c \in C$

$$\langle \mathbf{T}^C, V \rangle \models \phi[g(c), c];$$

and ϕ is utterance valid on \mathbf{T}^C iff ϕ is locally utterance valid on all contextual variants $\mathbf{T}^{C'}$ of \mathbf{T}^C . We say that ϕ is valid, locally utterance valid, or utterance valid respectively iff for all contextualised frames \mathbf{T}^C ϕ has the variety of validity in question. □

This definition needs some explanation. The basic distinction being drawn is between plain validity and the two varieties of validity involving the word ‘utterance’. This type of distinction dates back to at least 1968 and is essentially that drawn by Richard Montague between pragmatic validity and ordinary validity [68, page 107]. Montague noted that while certain expressions involving indexicals such as ‘I exist’ were not ‘logically’ or ‘semantically’ valid, the whole point about the way indexicals function was that the utterance of such an expression was a guarantee of its truth. This is an interesting property, and Montague introduced the term ‘pragmatic validity’ for it. ⁷

⁷The distinction is fundamental in the Californian tradition: for example David Kaplan introduces the notion of ‘validity’ [56, page 408], which is essentially what we have termed above ‘local utterance validity’; and ‘neotraditional validity’ [56, page 410], which is the analog of our ‘validity’.

Now the definition of validity simpliciter just given is unproblematic; the quantification across all pairs $[t, c]$ captures the ‘logical’ or ‘semantical’ notion required. Note that it is a corollary of the previous lemma that for any frame \mathbf{T} , any c-frame \mathbf{T}^C based on \mathbf{T} , and any wff ϕ , ϕ is valid on \mathbf{T}^C iff ϕ is ordinary TREF valid on the underlying frame \mathbf{T} of \mathbf{T}^C .⁸

Pinning down a satisfactory ‘pragmatic’ notion of validity is rather more interesting. Local utterance validity is an obvious first attempt to capture such a notion, and as it only quantifies across pairs of the form $[g(c), c]$ — that is, across those pairs where ‘something is actually said’ — it is fundamentally along the right lines. Nonetheless, the distinction it draws is not subtle enough. \mathcal{L}_c^0 is a particularly simple contextualised language, and none of the clauses in its truth definition makes any essential reference to context. Because of this insensitivity to context, any proposed distinction between ‘semantical’ and ‘pragmatic’ validity on c-frames ought to coincide for \mathcal{L}_c^0 . However local utterance validity does *not* coincide with validity. To take a simple example, consider any contextualised frame $(\mathbf{Q} \oplus \mathbf{Z})^C$ whose underlying frame $\mathbf{Q} \oplus \mathbf{Z}$ consists of a copy of the rational numbers \mathbf{Q} followed by a copy of the integers \mathbf{Z} , and whose contexts C are all mapped by g into points in \mathbf{Q} . Because contexts are all located on the rationals, contexts all occur in a region of density, and thus $F\phi \rightarrow FF\phi$ is locally utterance valid. But clearly this formula is not valid as it can be falsified on the discrete region \mathbf{Z} . In short, local utterance validity is weaker than validity for \mathcal{L}_c^0 , and the difference noted has nothing to do with issues we would term pragmatic; it reflects local structural differences of the underlying frame.⁹ We should abstract from these effects, and this is achieved in the definition of utterance validity by quantifying over all contextual variants \mathbf{T}^C of

⁸To see the left to right implication, argue by contrapositive. Suppose for some valuation V and point $t \in T$ that $\langle \mathbf{T}, V \rangle \not\models \phi[t]$. Choose any $c \in C$ and let V^C be any \mathcal{L}_c^0 valuation such that $V^c(a, c) = V(a)$ for all atoms a . Then by the previous lemma $\langle \mathbf{T}^C, V^C \rangle \not\models \phi[t, c]$, the desired result. The reverse implication is similar.

⁹More precisely, local utterance validity fails to coincide with validity in this example because the underlying frame $\mathbf{Q} \oplus \mathbf{Z}$ is not *homogeneous*. van Benthem [5, page 39] defines this as follows:

\mathbf{T}^C . The stronger notion of utterance validity draws the distinction we are interested in more accurately, and it is straightforward to show that an \mathcal{L}_c^0 formula ϕ is valid iff it is utterance valid, just as we would hope. In the stronger languages we shall later consider these two notions will be distinct, and the divergence will be for genuinely pragmatic reasons.

Before introducing our first properly context sensitive language let us briefly compare the contextualisation strategy being adopted here with that of three major papers in the Californian tradition from which the ideas of this section ultimately stem: Kamp's "Formal Properties of 'Now'" [52], Kaplan's "On the Logic of Demonstratives" [56], and Richard Montague's "Universal Grammar" [69].

Kamp does not include a set of contexts. He does evaluate wffs with respect to a pair of indices — and indeed this paper was the first to show how useful this 'double indexing' strategy could be — but both elements of the pair are times. As with our semantics, however, one temporal component is the timestream, and is used in defining the semantics of the tense operators; while the other 'lies idle' for this part of the language, and comes into its own (as will our contexts), when treating such indexicals as 'Now'. In effect, Kamp is using c-frames which have the form $\langle T, <, T, id_T \rangle$, where id_T is the identity function on T . But while the difference is technically trivial, it's precisely the isolation of this special case that opened the way for the symmetrical treatment of the two indices that became routine in two dimensional logic. Conceptually the two indices are distinct: one is a context, the other a time, and should be exploited differently. In passing, Kamp did think of the second time as recording time 'utterance time': see [52, page 238], and his exploitation of this second index is 'clean'.

The presentation here is essentially Kaplanian. In particular, the idea of using a function to assign an utterance time to an independent set of contexts is his. Conceptually

a frame \mathbf{T} is homogeneous iff for any $t, t' \in T$ there is an order automorphism α on \mathbf{T} such that $\alpha(t) = t'$. It is easy to show that if ϕ is locally utterance valid on some c-frame \mathbf{T}^C based on a frame \mathbf{T} that is homogeneous in this sense, then ϕ must also be valid on \mathbf{T}^C .

this is the crucial move: it firmly underlines that the second index is intended to be a context of utterance, not just another time we can treat as we please. Although the change seems trivial, as we shall see its effects run deep.

Nonetheless, although the basic machinery is Kaplanian, the presentation I have chosen is more akin to Montague's [69, page 228]. There is nothing deep about this choice; it's merely that I wanted to avoid discussing such matters as 'character' and 'content'. While Kaplan's distinction was historically influential, I don't find this terminology particularly useful.

This then is our base language \mathcal{L}_c^0 , a language of TREF interpreted on frames augmented by a notion of context. The next step is to exploit these contexts in a fashion that reflects the workings of indexicals in natural language. This will be done by sorting.

7.2.2 The language \mathcal{L}_c^1

We now introduce the simplest extension of \mathcal{L}_c^0 , the language \mathcal{L}_c^1 . This language is \mathcal{L}_c^0 extended by the addition of a new sort of atomic symbol, a sort with only one element which we write as *now*. All four sorts of \mathcal{L}_c^1 — VAR, NOM, INOM and $\{now\}$ — are pairwise disjoint and the wffs of \mathcal{L}_c^1 are constructed in the usual TREF fashion from this base.

now is intended to be a 'name for now', which given our sorted atomic sentence strategy means that we want it to be a special atom which in any context c is true at the 'now time' or 'utterance time' of c , and true at no other time. To accomplish this we interpret \mathcal{L}_c^1 on c -frames $\langle T, <, C, g \rangle$ as follows: we demand of any function $V : ATOM_{\mathcal{L}_c^1} \times C \rightarrow Pow(T)$ if it is to count as an \mathcal{L}_c^1 valuation that not only must it satisfy the constraints obeyed by the \mathcal{L}_c^0 valuations, but in addition that $V(now, c) = \{g(c)\}$, for all $c \in C$. We define an \mathcal{L}_c^1 model \mathbf{M}^1 to be pairs $\langle \mathbf{T}^C, V \rangle$, where \mathbf{T}^C is a c -frame and V an \mathcal{L}_c^1 valuation on \mathbf{T}^C ; and evaluate \mathcal{L}_c^1 wffs at pairs $[t, c]$ in \mathcal{L}_c^1 models by following the recipe specified in the previous section for \mathcal{L}_c^0 wffs. In short, we have a new (extended) concept of valuation, but once this has been specified all else remains

unchanged, a pattern repeated in the remaining languages that will be introduced in this section.

In any context *now* is assigned a singleton subset of the frame, namely $\{g(c)\}$, thus *now* is a nominal. The important point of course is *which* singleton subset is assigned: unlike ordinary nominals which in any context ‘freely refer’, *now* is highly constrained. Note the ‘direction of constraint’ — it lies along the contextual dimension. If you like, when we were working in ordinary (uncontextualised) tensed languages what we did was sort the denotation of atoms on the temporal line, creating in the process such referential sorts as nominals and interval nominals. What we are doing now is to sub-sort the referential sorts by exploiting the additional structure provided by C and g .

Let us examine some examples of \mathcal{L}_c^1 validities and utterance validities. Firstly some plain validities. Assume that all the frames underlying our c-frames are STOs. As these frames are all irreflexive, $now \rightarrow \neg Fnow$ is valid, as is its mirror image $now \rightarrow \neg Pnow$. Similarly $now \rightarrow GPnow$, $now \rightarrow G(Fnow \rightarrow now)$ and $FFnow \rightarrow Fnow$.

What about utterance validity? As the semantics of *now* is clearly bound up with the context of utterance we would hope that we do have some genuinely ‘pragmatic’ validities. We do: *now*! Clearly *now* is not valid, so our two notions of validity differ in \mathcal{L}_c^1 . Further note that if ϕ is valid, ϕ must also be utterance valid, so all the examples of validity just noted in the previous paragraph are also utterance valid. But then, by modus ponens, we have that $\neg Fnow$, $\neg Pnow$, $G(Fnow \rightarrow now)$ and $GPnow$ are all utterance valid. None of these wffs is valid. Note that the conjunction of these first two wffs amounts to saying that any contextualised observer knows that now occurs right now, and not before and not after.

The introduction of *now* allows us to mimic in our tensed languages what we might term the ‘canonical use’ in natural language of the word now.¹⁰ What is this canonical use? Simply, to refer to the time of utterance. As Hans Kamp pointed out in “Formal

¹⁰This cheerful use of the word ‘canonical’ glosses over many difficult issues. The uses of ‘now’ in natural language are diverse and subtle.

properties of ‘now’ ” [52], there is a temptation to think that the function this word performs is vacuous — surely “It is raining” has exactly the same truth conditions as “It is now raining”? In this case the ‘now’ is redundant, but Kamp gives two examples which show that ‘now’ can have a genuine semantic role:

- (1) I learned last week that there would be an earthquake.
- (2) I learned last week that there would now be an earthquake.

Clearly the ‘now’ serves to make the second sentence more temporally specific than the first, so ‘now’ in general is not vacuous in natural language. Let us consider another pair of examples, also due to Kamp [52, page 231]:

- (1) A child was born that would be ruler of the world.
- (2) A child was born that will be ruler of the world.

The first can be translated into quantified tense logic by

$$P(\exists x)(x \text{ is born} \wedge F(x \text{ is ruler of the world}))$$

The second has no such transcription. The problem lies with the ‘will’; this forces the time when the child assumes its role to occur *after* the point of speech. This effect cannot be coded in orthodox quantified tense logic. Kamp introduces his Now operator N to cope with this. Essentially this operator works by ‘remembering’ the time of utterance¹¹ and whenever a wff of the form $N\phi$ is encountered, ϕ is evaluated at this time. That is, N forces a jump to the time of utterance and evaluates ϕ there. This enables Kamp to transcribe the second sentence, by:

$$P(\exists x)(x \text{ is born} \wedge NF(x \text{ is ruler of the world})).$$

¹¹Our $g(c)$. As has already been mentioned the formalities of Kamp’s system are different from those of \mathcal{L}_c^1 , but the semantics of the N operator essentially involves keeping track of $g(c)$. In the notation van Benthem uses in a paper we will later resort to frequently [4, page 412], the Kamp style semantics of N is given by $\mathbf{M} \models N\phi[t_0, t]$ iff $\mathbf{M} \models \phi[t_0, t_0]$. That is, N is a diagonalisation operator.

In \mathcal{L}_c^1 we would write this as

$$P(\exists x)(x \text{ is born} \wedge L(now \rightarrow (F(x \text{ is ruler of the world}))))^{12}$$

As this last example suggests, Kamp's calculus can be simulated in \mathcal{L}_c^1 . We can define a now operator in \mathcal{L}_c^1 by $N\phi =_{def} L(now \rightarrow \phi)$. We have decomposed Kamp's operator into a 'shift' and a 'referral'.

However Kamp's calculus cannot simulate \mathcal{L}_c^1 — indeed it cannot even simulate the fragment of \mathcal{L}_c^1 consisting of all and only the wffs made using variables and *now*. The clue is Kamp's 'Eliminability Theorem'.¹³ This theorem shows that for the propositional part of the Now calculus, every formula containing occurrences of the *N* operator is equivalent to a formula not containing such occurrences: *N* is eliminable. But this means that irreflexivity is not definable in Kamp's calculus — no purely priorean formula defines this condition, and every *N* containing formula is equivalent to such a wff. On the other hand we can define irreflexivity in the $VAR \cup \{now\}$ fragment of \mathcal{L}_c^1 ; $now \rightarrow \neg Fnow$ suffices.

In the paper "Now"[80, pages 112–113] Prior very briefly introduces a name for 'now' in precisely the fashion we have been discussing in this chapter. However, after toying with the idea for a couple of paragraphs he drops the idea in favour of taking Kamp's operator as primitive. He doesn't seem to have observed that the two approaches

¹²Actually, we wouldn't. Adhering to our 'the simple past refers' motto we would represent it by

$$P(i \wedge (\exists x)(x \text{ is born} \wedge L(now \rightarrow (F(x \text{ is ruler of the world}))))).$$

where the *i* picks out the time of birth.

¹³This is what Burgess terms it [16, page 121]. Kamp's original proof may be found in [52, page 251].

are not equivalent.¹⁴ We'll later see reasons for preferring the use of a referential sort for 'now', to Kamp's diagonalisation strategy.

7.2.3 The language \mathcal{L}_c^2

Let us now consider an extension of \mathcal{L}_c^1 that mimics the adverbials 'yesterday', 'today' and 'tomorrow', the language \mathcal{L}_c^2 . \mathcal{L}_c^2 has a fifth atomic sort in addition to those of \mathcal{L}_c^1 , the sort

$$CDA = \{yesterday, today, tomorrow\}.$$

CDA stands for *contextualised day adverbials*. We select *yesterday*, *today*, and *tomorrow* so that they are distinct from all the \mathcal{L}_c^1 atoms, and then define the $ATOM_{\mathcal{L}_c^2} = ATOM_{\mathcal{L}_c^1} \cup CDA$. The \mathcal{L}_c^2 wffs are then formed in the usual way from these atoms.

Clearly *yesterday*, *today* and *tomorrow* are going to be interval nominals contextually constrained in some fashion; but in fact we are going to have to introduce some more structure for these adverbials as they don't denote just any interval, they denote *days*. That is, the use of their natural language correlates presupposes a view of time rich enough to support a 'day structure': a division of the temporal flow into equal length time periods such that every point of time occurs on a unique day and such that the sequence of days is a discrete structure; and further such that each day contains many (if not infinitely many) points of time. Moreover, once we consider days it's natural to move on and model 'calendar terms' which, at least in western traditions, seem to presuppose that time is linear. In short, in order to give an adequate model of these adverbials and dates we are going to have to restrict ourselves to a class of linear frames rich enough to support a day structure.

¹⁴Jon Oberlander earlier noted a weaker version of this result as part of his comparison of Kaplan's Logic of Demonstratives and the system IQ. Details of his result may be found in [73] or [71].

Now we could just straightaway opt for one of the ‘scientific’ structures of time, say \mathbb{R} or \mathbb{Q} , but doing so seems rather silly; it overstructures our temporal ontology.¹⁵ Our intuitive notions of time are rich — the concept of a ‘day’, for example, is sophisticated — but they are ‘fuzzy’ as regards many mathematically important choicepoints such as that between discreteness and continuity. So let’s try and isolate a class of frames rich enough to support a ‘day structure’ that does not impose such choices — or at least, let’s try to minimise our impositions. The class of *Q-containing* frames is my choice, at least for the purposes of this section.

A frame $\mathbf{T} = \langle T, <_t \rangle$ is called *Q-containing* iff \mathbf{T} is a strict total order and there is an injection $f : \mathbb{Q} \rightarrow T$ that isomorphically embeds \mathbb{Q} in \mathbf{T} in the following fashion: for any $t \in T$ there are $q, q' \in \mathbb{Q}$ such that $f(q) <_t t <_t f(q')$. This last condition guarantees that \mathbb{Q} is ‘indefinitely mingled’ in \mathbf{T} : there are no points $t \in T$ greater than (or less than) every point in the image of f — no part of \mathbf{T} ‘sticks out of the end’ of the image of \mathbb{Q} . Some examples of *Q-containing* frames are \mathbb{Q} itself, \mathbb{R} and the structure $\mathbb{Q} \odot \mathbb{Z}$.

Before defining the obvious ‘day structure’ on such frames, some conventions. Firstly, given a *Q-containing* frame \mathbf{T} we act as if $\mathbb{Q} \subseteq T$. Thus we treat certain elements $q \in T$ — namely, the image points of f — as if they actually were rational numbers and add and subtract them. There is a certain sloppiness in doing this: given a *Q-containing* frame \mathbf{T} there may be many distinct injections f that isomorphically embed \mathbb{Q} in \mathbf{T} in the required fashion, and each f gives rise to a different set of ‘rationals’ in \mathbf{T} ; but we’ll ignore this and pretend that we know which of the points in any *Q-containing* frame are rationals. Indeed we’ll act as if $\mathbb{Z} \subseteq \mathbb{Q}$ and treat certain elements of such frames as integers.

We now impose a day structure on such frames. We define a set of days and three functions: a next day function, a previous day function, and a function that answers the question ‘What day is it?’. Let \mathbf{T} be a *Q-containing* frame. Firstly the days on \mathbf{T} are

¹⁵See [4, page 404].

defined to be:

$$D = \{[z, z + 1) : z, z + 1 \in Z\}.$$

As usual by the half-open interval $[z, z + 1)$ is meant $\{t \in T : z \leq_t t <_t z + 1\}$. Clearly $\bigcup D = T$, and moreover every point $t \in T$ occurs in a unique day. The successor and predecessor functions on days of \mathbf{T} are both functions with domain D and range D and are given by:

$$\begin{aligned} \text{next}([z, z + 1)) &= [z + 1, z + 2) \\ \text{prev}([z, z + 1)) &= [z - 1, z); \end{aligned}$$

and the ‘What’s today?’ function on \mathbf{T} is the function *day* with domain T and range D defined by:

$$\text{day}(t) = [z, z + 1) \in D \text{ such that } t \in [z, z + 1).$$

Putting all this together, given a \mathbf{Q} -containing frame \mathbf{T} , by the day structure \mathbf{D}^T on \mathbf{T} is meant the 4-tuple

$$\langle D, \text{next}, \text{prev}, \text{day} \rangle$$

where D is the set of days on \mathbf{T} , and *next*, *prev* and *day* are the functions just defined.

We now interpret \mathcal{L}_c^2 on c-frames that carry a day structure. That is, we interpret \mathcal{L}_c^2 on structures of the form $\langle \mathbf{T}, C, g, \mathbf{D}^T \rangle$, where \mathbf{T} is a \mathbf{Q} -containing frame, $\langle \mathbf{T}, C, g \rangle$ is a c-frame, and \mathbf{D}^T is the day structure on \mathbf{T} . As usual we use constrained valuations to interpret our various sorts of atoms. \mathcal{L}_c^2 valuations are functions

$$V : \text{ATOM}_{\mathcal{L}_c^2} \times C \longrightarrow \text{Pow}(T)$$

that satisfy all the constraints placed on \mathcal{L}_c^1 valuations, and in addition such that:

$$\begin{aligned} V(\text{yesterday}, c) &= \text{prev}(\text{day}(g(c))) \\ V(\text{today}, c) &= \text{day}(g(c)) \\ V(\text{tomorrow}, c) &= \text{next}(\text{day}(g(c))), \end{aligned}$$

for all contexts c . As the presence of the g in the above definitions makes clear, *yesterday*, *today* and *tomorrow* are being treated as indexicals.

As usual, these atomic level semantic stipulations are all we need to state: compound wffs are treated in the usual way. That is, the definition of an \mathcal{L}_c^2 model, and truth in such a model at a pair $[t, c]$ is defined as for \mathcal{L}_c^0 and \mathcal{L}_c^1 .

With all this to hand, what can we do? First let's encode "John ran yesterday", "John ran today" and "John will run today" using our heuristic of decomposing the tense into a shift and a referral. In the first sentence the tense is past, thus a backwards shift is required and hence we use P . Equally clearly the function of 'yesterday' is to give more precise information about which past time is being referred to, so we should fill the referral slot by 'yesterday'. This yields $P(\text{yesterday} \wedge \text{John runs})$. In similar fashion we may translate the other two sentences to $P(\text{today} \wedge \text{John runs})$ and $F(\text{today} \wedge \text{John runs})$. Now these translations 'work' — that is, they give the right truth conditions — but are there any reasons for finding the use of \mathcal{L}_c^2 particularly interesting? I think two considerations suggest we are on the right track.

Firstly, the use of \mathcal{L}_c^2 seems to have advantages over the use of 'Yesterday', 'Today' and 'Tomorrow' operators. The use of such operators has been criticised on the grounds that neither of the scoping possibilities they permit allow adequate encodings even of such simple sentences as those just given.¹⁶ That is, neither $PY(\text{John runs})$ nor $YP(\text{John runs})$ is felt to be adequate to represent the first sentence. Our encodings don't have this problem.

The second reason is more interesting. There are well known sentences where the

¹⁶To take a recent example, Eleonore Oversteegen writes:

if temporal adverbials like *yesterday* are treated as operators, there is a scope problem: both sequences, *yesterday* in the scope of the tense operator and the tense operator in the scope of *yesterday* are obviously wrong. [74, page 3].

See also Barbara Partee's remarks [75, page 257].

‘tense shift’ clashes with the ‘referral’. Two simple examples are:

*John will run yesterday.

*John ran tomorrow.

Finding representations that handle such examples correctly has proved tricky — in fact Harper and Charniak [44] list the ability to handle such examples as one of their five desiderata for any system of temporal representation.¹⁷ Their representations in \mathcal{L}_c^2 are $F(yesterday \wedge \text{John runs})$ and $P(tomorrow \wedge \text{John runs})$ respectively. Clearly neither wff is satisfiable in any \mathcal{L}_c^2 model. To some degree \mathcal{L}_c^2 is successfully mimicking the way temporal reference and tense interact in English.

7.2.4 The language \mathcal{L}_c^3

\mathcal{L}_c^3 is an extension of \mathcal{L}_c^2 that models the ‘calendar terms’ of natural language. In addition to the sorts of \mathcal{L}_c^2 it has five more sorts. These are:

<i>DAY</i>	=	$\{Sunday, \dots, Saturday\}$
<i>DATE</i>	=	$\{1, \dots, 31\}$
<i>MONTH</i>	=	$\{January, \dots, December\}$
<i>YEARS</i>	=	$\{1, 2, 3, \dots, 1989, 1990, \dots\}$
<i>ERA</i>	=	$\{BC, AD\}$

We define *CAL* to be $DAYS \cup DATE \cup MONTH \cup YEAR \cup ERA$ — we assume of course that all these sets are mutually disjoint from each other and from the sorts of \mathcal{L}_c^2 — and then define $ATOM_{\mathcal{L}_c^3} = ATOM_{\mathcal{L}_c^2} \cup CAL$. The \mathcal{L}_c^3 wffs are made in the usual fashion from these atoms. As an example of how we hope to use this language, the sentence “John ran on Monday 4th December 1989” will be represented by:

$$P(Monday \wedge 4 \wedge December \wedge 1989 \wedge AD \wedge \text{John runs}).$$

¹⁷Jon Oberlander’s work first drew my attention to these types of clashes. For his solution to the problems they pose, couched in the language IQ, see [72] or [71].

Before sketching the semantic machinery, a remark. Note that with the exception of the atoms in *ERA* — and the atom 1 of *YEARS* — none of these atoms are interval nominals as they either ‘denote periodically’ or denote two intervals. ‘Tuesday’ for example must denote a day every seven days. In general, we might describe all the elements in *DAY*, *DATE* and *MONTH* as ‘periodic interval nominals’. ‘Binary interval nominals’ seems an appropriate label for the atoms in *YEARS* save 1.

The semantics of \mathcal{L}_c^2 was based on 4-tuples of the form $\langle \mathbf{T}, C, g, \mathbf{D}^T \rangle$, where \mathbf{T} was a Q-containing frame. We extend these structures to 5-tuples of the form $\langle \mathbf{T}, C, g, \mathbf{D}^T, \text{Cal}^D \rangle$ in order to give the semantics of \mathcal{L}_c^3 . The new component is called a *calendar structure*, and as the superscript on Cal^D is intended to indicate, its definition will make use of the day structure \mathbf{D}^T . Cal^D is a 4-tuple:

$$\text{Cal}^D = \langle \text{Weeks}, \text{Months}, \text{Years}, \text{Eras} \rangle$$

Each of its four components, *Weeks*, *Months*, *Years* and *Eras*, is a partition of $T = \bigcup D$.

¹⁸ We place the obvious conditions on each of these partitions. Consider the case of *Years*, for example. Firstly, let a year_1 be a set containing 365 or 366 consecutive elements of D . ¹⁹ We then define a year_2 to be $\bigcup \text{year}_1$, for any year_1 . The element *Years* in any calendar structure Cal^D is then defined to be a ‘leap year structured’ partition of T such that every element of the partition is a year_2 . (By ‘leap year structured’ is meant that every fourth element in *Years* is a year_2 made from a year_1 containing 366 days, while the remaining three out of four year_2 are made up of year_1 containing only 365 days.)

We construct *Weeks*, *Months* and *Eras* in similar fashion. Every week_2 in *Weeks* is made up of the points in seven consecutive days; every month_2 in *Months* is made

¹⁸By a partition of a non-empty set X is meant a subset P of $\text{Pow}(X)$ such that $\bigcup P = X$, each $p \in P$ is non-empty, and for all $p, p' \in P$, $p \neq p'$ implies $p \cap p' = \emptyset$.

¹⁹Recall that we have both successor and predecessor functions on D , so ‘consecutive days’ can be defined in the obvious way.

up of the points in either 28, 29, 30, or 31 consecutive days; and *Eras* partitions T into two. In addition we must ensure that each of these four sets *Weeks*, *Months*, *Years* and *Eras* ‘hangs together’ correctly. For example, the end of the first era should coincide with the end of some year; the start of a year should coincide with the start of a month containing 31 days, while the second month in any year should contain 28 days, unless the year is a leap year when it must contain 29; and so on — the details can be left to the reader’s ‘calendar knowledge’. We assume without further ado the natural induced ordering of *Weeks*, *Months*, *Years* and *Era*. For example, if $m, m' \in \text{Months}$ we say that $m < m'$ iff $\forall t \in m \forall t' \in m' (t < t')$.

We now describe \mathcal{L}_c^3 valuations. These will link \mathcal{L}_c^3 with our structures $\langle \mathbf{T}, C, g, D^T, \text{Cal}^D \rangle$ and are functions

$$V : \text{ATOM}_{\mathcal{L}_c^3} \times C \longrightarrow \text{Pow}(T)$$

which obey the constraints that typify \mathcal{L}_c^2 valuations, and in addition constraints concerning the sorts in *CAL*. Clearly the constraints on these new sorts will be stated in terms of the additional structure Cal^D .

I’m only going to deal fully with the two simple atoms in *Era*. Firstly recall that *Era* is a binary partition of T ; let us write it as $\{bc, ad\}$, where $bc < ad$. We demand that functions V , if they are to be \mathcal{L}_c^3 valuations, must be such that:

$$\begin{aligned} V(BC, c) &= bc \\ V(AD, c) &= ad \end{aligned}$$

for all $c \in C$. Note that these atoms are not indexicals. Essentially we have said that V must ‘ignore context’; both *BC* and *AD* rigidly designate the obvious eras no matter what context they are uttered in.

As to the other sorts, let *day*, *date*, *month*, and *year* be metavariables over the atoms of *DAY*, *DATE*, *MONTH* and *YEAR* respectively. Then we demand of any

function V with aspirations to valuationhood that:

$$\begin{aligned} V(\text{day}, c) &= S, \text{ where } S \subseteq D^T \\ V(\text{date}, c) &= S, \text{ where } S \subseteq D^T \\ V(\text{month}, c) &= S, \text{ where } S \subseteq \text{Months} \\ V(\text{years}, c) &= S, \text{ where } S \subseteq \text{Years} \end{aligned}$$

for any $c \in C$. I'm not going to attempt to fill in the details — I'll merely note that doing so is not quite as dull as it may appear. Firstly, note that the atoms are *mutually* constrained in their interpretation. For example, given the denotation of any year, we know it contains the denotations of twelve different months. This immediately takes us beyond the simple definition of sorting given in the previous chapter. Also note that calendar terms cover frames in a fashion reminiscent of strong nominals, by means of periodicity. In fact periodicity seems an interesting intermediary between the covering of frames by totally independent strong nominals, and the 'structured term covering' of languages of arithmetic.

With \mathcal{L}_c^3 we have a system with a fairly wide repertoire of referential sorts. Other extensions are routine. For example, we could add 'clock terms' to the language, and define 'clock structures' in a manner that would allow "John reached the summit at three o'clock" to be represented by

$$P(\text{three o'clock} \wedge \text{John} \dots \text{summit}).$$

7.3 Then and now: some history

In this section we revert to the discussion of TREF and Reichenbach. We will learn more about the modeling capabilities of TREF, but our concerns are primarily methodological. We will examine the main line of development of two dimensional logic and find it wanting. In order to appreciate why, we need to consider anew what we have done in this chapter.

TREF can plausibly be regarded as a model of Reichenbach's ideas in a Priorean framework. Importantly, double indexing is not used in TREF semantics. While later we

did introduce such ideas, in a Kaplanian or contextualised form, to allow certain atoms (*now*, *yesterday*, *today* and *tomorrow*) to have special referents — double indexing was not used to ‘simulate reference’. As always throughout the thesis, sorting was the referential mechanism.

I am emphasising these points because there were earlier attempts to model Reichenbach in tense logic, and these attempts crucially depended on the use of a doubly indexed semantics exploited by powerful, and in some cases unintuitive, two dimensional operators to simulate reference. In fact, as far as the development of ‘double indexed’ *logics* is concerned, these attempts have set the agenda for most subsequent work. The reason is this. The first attempt to extend the use of two dimensionality to model reference beyond Kamp’s modeling of ‘now’ was Vlach’s modeling of ‘then’. Vlach did this by introducing an operator with a number of interesting properties — it’s the other diagonalisation operator. Subsequent logical developments concentrated on the properties of these operators, and a number of more complex ones later introduced to model Reichenbachian ideas.

As was stated in the introduction, the effect of these developments on the application of tense logic in natural language semantics has been largely malign; it’s doubtful whether many enthusiastic advocates for its use could be found nowadays. This seems due at least in part to the direction the extensions discussed below have led.²⁰ Some of the later extensions have little — if anything — to recommend them at the intuitive level. They succeed in simulating reference, but the negative connotations of ‘simulate’ apply here: the results are exercises in ‘clever programming’.

The discussion that follows will be largely critical of this tradition, but the criticism has a point. “Tense Logic and Standard Logic” by Johan van Benthem [4] is the most

²⁰There are other reasons. One is that tense logic (like classical logic) has always been traditionally preoccupied with sentence level phenomena, while much current research on temporal semantics is concerned with the discourse level.

searching critique of the aspirations of tense logic I know of.²¹ One of its major themes is this: if one of Priorean tense logic's virtues is simplicity, this virtue is quickly lost when devices to cope with further aspects of natural language are added — indeed quite quickly the use of classical languages as a modeling formalism seems simpler. In fact, as the paper notes, the two dimensional formalisms surveyed below lead with seeming inexorability, to 'intensional languages' that appear to be merely notational variants of Quinean variable-free formulations of first order languages. Once this stage has been reached, or neared, the natural question to ask is "Why on earth use tense logic? Why not use a classical language straight off and have done with it?"

My answer is that the uses the sorted atomic sentence strategy have been put to so far have been extremely simple, and intuitively revealing — we don't seem to be embarking on the path that van Benthem condemns. However simplicity is partly a matter of taste, and it would be nice to be a little more objective. In this section we are going to show that two of the natural language examples which motivated first the addition of Vlach's operator, and later the introduction of a countably infinite sets of N operators and Vlach operators, can already be represented in TREF. This is not difficult (the corresponding wffs can be written down as easily as first order representations), and for TREF we have a complexity measure: we're not above D logic in the expressibility hierarchy. Actually, I suspect that the languages \mathcal{L}_c^1 to \mathcal{L}_c^3 are also simple, but as we shall see later in the section, this is definitely something that needs proof — hence the usefulness of the purely TREF examples. In short, I don't believe that van Benthem's criticisms can be applied to TREF — and this for both intuitive and technical reasons. I further believe that the series \mathcal{L}_c^1 , \mathcal{L}_c^2 and \mathcal{L}_c^3 escapes similar censure, though (for the time being) my defense relies purely on the grounds of their naturalness. Of course none of this establishes that we *should* use such languages — but positive arguments in their favour must await the conclusions of the thesis. The following discussion is closely based around van Benthem's

²¹It's certainly the most interesting: many papers, both for and against the use of tense logic, predate such logical developments as correspondence theory, and now read like reports of distant (perhaps apocryphal) battles concerning angels and pinheads.

presentation on [4, pages 412 – 428], and his definitions of these operators are used in what follows.

Consider the sentence “One day, all persons alive now will be dead”. Like the example considered earlier this can be represented with the help of Kamp’s operator by means of:

$$F\forall x(N\text{Alive}(x) \rightarrow \text{Dead}(x)),$$

but not by any purely Priorean formula. Frank Vlach [109] noted that the past tensed version of this sentence, “One day, all persons alive then would be dead”, required even more equipment. Simply affixing a P to the previous wff, the Priorean recipe for forming past tenses, does not yield a suitable representation; and in fact no Kampian formula will suffice. “As a true Priorean”, writes van Benthem, “Vlach introduces a new operator K ” [4, page 416]. In van Benthem’s notation the semantics of this operator is given by:

$$\mathbf{M} \models K\phi[t, t_0] \text{ iff } \mathbf{M} \models \phi[t, t].$$

K takes the other route to the diagonal and forces evaluation there. It’s hard to express the semantics of the Vlach operator in Kaplanian semantics,²² but intuitively K ‘moves now’. ‘Pretend until further notice that the present point of evaluation is now’, is a rough gloss. Using Vlach’s operator the previous sentence can now be represented:

$$PKF\forall x(N\text{Alive}(x) \rightarrow \text{Dead}(x)).$$

As we shall shortly see, Vlach’s addition is not enough to cope with the demands of natural language, but this is the least of our concerns: Vlach’s ideas are incompatible with the contextual ideas of the California theory of reference. Adding the Vlach operator effectively changes the meaning of the N operator: we can now ‘overwrite’ the value N forces evaluation at. Whatever N means in the richer language, it doesn’t mean ‘now’.

²²How hard depends on how many simplifying assumptions one is willing to make, and how far one is prepared to tinker with the contextual semantics. The N diagonalisation operator makes sense in a Kaplanian framework; it’s rather doubtful that K does. (See the later discussion of Aqvist’s *id.*)

It may be objected that it never did mean this, and to some extent (as I prefer the sorted approach using *now*) I agree with this. Nonetheless, *N* makes sense in Kaplanian semantics, whereas the addition of the Vlach operator is incompatible with further contextual aspirations.

Note that from the contextual perspective Vlach's attempt to use double indexing machinery to model 'then' is misguided. 'Then' is not context sensitive in the way that 'now' or 'today' are. Its referent does depend on contextual features, but is not calculable as a simple function of utterance time. It's a word that typically accompanies either an act of temporal deixis (for example, when watching a historical film) or temporal anaphor (when telling a story), not a true indexical. Using the simple contextual apparatus of the Kaplanian approach to simulate such reference is misleading.

The introduction of the Vlach operator bifurcated the concept of 'double indexing': from then on there was the original contextual sense, and the newer two dimensional one. Indeed the term 'two dimensional' is an excellent description of the branch that came to dominate in formal work. In Vlach's system the two indices are treated on a par. They're both times, there is no conceptual distinction between them any more, and we are free to introduce operators to manipulate them as we please. It's a relatively short step from here to the use of multiple indices and arbitrary manipulation operations, and the step was shortly taken.

Note that the above example can easily be represented in TREF using our 'shift and refer' heuristic:

$$P(j \wedge F\forall x(L(j \rightarrow \text{Alive}(x)) \rightarrow \text{Dead}(x))).$$

The tense gives us a point of reference (here labelled by *j*), tells us where to look for it (backwards), and the subsequent use of 'then' shifts our attention to the people alive at this time (the construction $L(j \rightarrow \text{Alive}(x))$). If we liked, we could compress this last complex into a subscripted 'then' operator:

$$P(j \wedge F\forall x(\text{Then}_j \text{Alive}(x) \rightarrow \text{Dead}(x))).$$

In passing, I think the use of an interval nominal here would be better:

$$P(e \wedge F\forall x(\text{Then}_e \text{Alive}(x) \rightarrow \text{Dead}(x))),$$

and more generally I think such examples show that Reichenbach's 'point of reference' should be thought of as an interval.

Vlach's addition doesn't suffice to cope with the vagaries of natural language temporal reference. Consider the following sentence: "There will always jokes be told that were told at one time in the past." This is not representable (at least in its 'strong' reading) with Vlach's apparatus. In TREF it is represented in the obvious way: if a unique time is wanted, grab it with a nominal (or an interval nominal):

$$G\exists x(\text{Uttered Joke}(x) \wedge L(i \rightarrow \text{Uttered joke}(x))) \wedge Pi.$$

Vlach's response to such examples was to propose adding a collection of new Now operators $N_1, N_2, N_3 \dots$; and new K_1, K_2, K_3, \dots ; observing that this would take care of such anomalies. It would, but the objections are obvious. I'll content myself with the following one. The introduction of multiple N s merely as a counterbalance to the multiple Vlach operators removes the remaining (apparent) vestiges of contact with the Kaplanian motivation for two dimensional machinery. Such N s have little to do with 'now'; their status is merely formal — though this had not yet been realised.

The first person to knowingly introduce purely formal two dimensional operators into a system intended to model natural language was Aqvist [1], who introduced an operator \Box_z into Kamp's system.²³ In van Benthem's (simplified) account of it, this operator works as follows:

$$M \models \Box_z \phi[t, t_0] \text{ iff } M \models \phi[t_0, t].$$

The reader who ponders the semantics of this operator in Kaplanian terms will probably agree with Aqvist's assessment that this is an operator "for which no independent reading is codified" [1, page 4]; certainly no contextual meaning is codified. Using this operator Aqvist was able to simulate Reichenbachian ideas of tense, but there is a price. As van Benthem notes, the system is more powerful than Until/Since logic [51], which makes it very powerful indeed.

²³The explorations of the earlier [99] were purely formal.

Aqvist's work opened the floodgates. A slightly later paper by Aquvist and Guenther [2] introduced a certain *four* dimensional system with a wide selection of permutation and substitution operators; later work by Gabbay suggested generalising the enterprise to evaluation at arbitrary finite sequences of points [30]. From the perspective of this thesis there is little point to these attempts: they are not perspicuous, they have lost all contact with contextual intuitions, and they have effectively turned tense logic into a notational variant of first order logic — and all this to model the Reichenbachian notion of tense that TREF so straightforwardly embodies. In “Logic of Time” van Benthem remarks of the developments we have noted that “tense logic has embarked on an extremely abstract course” [5, page 132]. It has, but it needn't.

This concludes the major argument of the section, and it's tempting to stop at this point and draw something like the following conclusions: that if we want to model reference in tense logic we should add reference (via sorting), not simulate it using two dimensional ideas; and that the interesting uses of double indexing are those easily formulable in a Kaplan style semantics. I think the latter conclusion is by and large sound, but we need to be careful what we mean by the first. Just as there are ‘contextually acceptable’ operators (N) and ‘contextually unacceptable’ ones (K), so there are contextually acceptable sorts, and contextually unacceptable ones. Referential sorting in a two dimensional setting — even if the use of ‘funny operators’ is eschewed — is not guaranteed to be a ‘safe’ process. Intriguingly, such sorting *was* introduced by Aquvist in the two dimensional tradition, albeit in a form that I would describe as contextually unacceptable. Technically, however, the effects of these new sorts are fascinating. I'll briefly review van Benthem's discussion of this work (using his notation) [4, page 418 – 422], and draw the appropriate moral for the enterprises of this chapter.

Aqvist introduced three propositional constants, *bf*, *id* and *af* defined by:

$$\begin{aligned} \mathbf{M} \models bf[t, t_0] & \quad \text{iff} \quad t < t_0 \\ \mathbf{M} \models id[t, t_0] & \quad \text{iff} \quad t = t_0 \\ \mathbf{M} \models af[t, t_0] & \quad \text{iff} \quad t_0 < t \end{aligned}$$

Roughly speaking, *bf* is a rather abstract name for before, *af* for after, and *id* is a sort of ‘sliding now’. Perhaps a better name for *id* would be ‘occupied’, for if we defined an

analog in the Kaplanian semantics it would probably be:

$$\mathbf{M} \models id[t, c] \text{ iff there exists a } c' \text{ such that } g(c') = t$$

(There's not a clear sense of diagonal in Kaplanian semantics.)

The addition of *id* yields a very powerful language. We can define the other two constants by means of *Fid* and *Pid* respectively, and then, trading on the linear nature of Aqvist's frames define the *Now* operator by:

$$N\phi =_{def} H(id \rightarrow \phi) \wedge (id \rightarrow \phi) \wedge G(id \rightarrow \phi).$$

It is this sort which then allows the *Until*/*Since* operators to be defined. For example:

$$S(\phi, \psi) =_{def} NP(\phi \wedge G(bf \rightarrow \psi)).$$

This example shows that referential sorting in a two dimensional setting can be an extremely powerful mechanism. It certainly adds weight to the evidence that if we want perspicuous and tractable logics that mimic aspects of natural language we should work in the Kaplanian tradition. Nonetheless, the above discussion of *id* should also make us extremely wary believing that sorting in the Kaplanian framework is inherently safe. At present I have no expressibility results for \mathcal{L}_c^1 through to \mathcal{L}_c^3 , and while I believe they're relatively constrained languages, this is something that will require detailed examination. I am investigating this matter, and looking for completeness and decidability results as well.

7.4 Referential sorting and discourse phenomena

At the beginning of the chapter we saw how to represent in TREF examples of the deictic uses of tense ("I didn't turn off the stove") and anaphoric uses, where the anaphor had a definite antecedent ("When John saw Mary she crossed the street"). These examples were taken from Partee's well known paper "Nominal and Temporal Anaphora" [75]. In the same paper Partee discusses three other temporal analogs of pronominal forms:

temporal anaphora with definite antecedents, bound variable forms, and sentences that may be temporal analogs of donkey sentences. Partee's discussion is couched in the language of (temporal) DRT [53] [54], and draws on earlier work by Hinrichs' [47]. In this section I will show that Partee's examples can be straightforwardly accommodated in TREF (or one of the \mathcal{L}_e languages). I will largely defer making any general comments until I have presented the examples, all of which (with the exception of the very first) are taken from Partee's paper.

Consider the sentence "The shutters were closed. It was dark." The tense in the second sentence (which is a *state* sentence) anaphorically picks out the time referred to by the tense of the first sentence (which is also a state sentence). We would represent this in TREF by

$$P(e \wedge \text{The shutters} \dots \text{closed}) \wedge P(e \wedge \text{It dark}).$$

Actually, this permits the times of shutter closure and darkness to be disjoint, which may be felt to be undesirable. (Partee insists on an 'overlapping' reading [75, page 255].) The non-overlapping sense can be captured as follows:

$$P(e \wedge L(e \rightarrow (\text{The shutters} \dots \text{closed}))) \wedge (P(e \wedge L(e \rightarrow \text{It dark}))).$$

Arguably the original representing wff captures the semantics, and the additional machinery is an explicit encoding of its implicatural force: namely that there really was an interval in the past throughout which both pieces of information held. At any rate, I'll always use the simpler form when interpreting successive state sentences in the following examples, and merely note that the stronger form is representable.

Things are more interesting when we have a sequence of two or more *event* sentences. Consider the discourse: "Mary woke up sometime during the night. She turned on the light." Representing this in TREF by

$$P(e \wedge \text{Mary wake} \dots \text{night}) \wedge P(e \wedge \text{Turn} \dots \text{light})$$

is clearly inadequate: this representation allows illumination to precede awakening. Using a nominal i in place of the interval nominal e does not improve matters: Mary would

have to move impossibly fast. In the above discourse the time referred to by the second tense is 'just after' the time picked out by the first tense. This temporal advance of the referential focus through the discourse is^a typical feature of many sequences of sentences involving event verbs. Roughly, event verbs advance the narrative focus in time. Stative verbs don't, but comment on the most recent interval. (Needless to say this is a oversimplification, but here my aim is only to deal with Partee's examples.) I'll defer saying anything about the 'just' in the 'just after' till the end of the section; however encoding the 'after' is simple. To make the representations as perspicuous as possible, define $After(e)$ to be $(\neg e \wedge P e)$. As usual e is a metavariable over interval nominals. Using this notation we can now represent the previous discourse by:

$$P(e \wedge \text{Mary} \dots \text{night}) \wedge P(d \wedge After(e) \wedge \text{Turn} \dots \text{light}).$$

As a second example consider the following discourse which mixes stative and event verbs: "John got up, went to the window, and raised the blind. It was light out. He pulled the blind down and went back to bed. He wasn't ready to face the day. He was too depressed." We can represent this as follows:

$$\begin{aligned} &P(e \wedge \text{John get up}) \\ &\wedge P(d \wedge After(e) \wedge \text{go to the window}) \\ &\wedge P(c \wedge After(d) \wedge \text{raise the blinds.}) \\ &\wedge P(c \wedge \text{It light out}) \\ &\wedge P(b \wedge After(c) \wedge \text{He pull blind down}) \\ &\wedge P(a \wedge After(b) \wedge \text{go back to bed}). \\ &\wedge P(a \wedge \text{He not ready to face the day}) \\ &\wedge P(a \wedge \text{He too depressed}) \end{aligned}$$

This representation merely amounts to 'shift and refer' coupled with 'advance the reference time when an event verb is encountered'.

Let's turn to the 'bound variable' case. Consider the following sentence: "Whenever Mary wrote a letter, Sam was always asleep." As a first approximation we might try

$$L(\text{Mary write a letter} \rightarrow \text{Sam asleep})$$

trading on the gloss mentioned in the previous chapter that $L(\phi \rightarrow \psi)$ means ‘Whenever ϕ , ψ ’. Clearly this is very bad. It’s good as a representation of “Whenever Mary writes a letter, Sam is always asleep”, where the present tense in conjunction with the temporal quantifier ‘whenever’ achieves the effect of abstracting from temporal placement completely, but in the previous example the quantification is clearly restricted to past times. The required representation is thus:

$$H(\text{Mary write a letter} \rightarrow \text{Sam asleep})$$

In short there is no referential effect (no nominals or interval nominals are required) and we can think of the effect of the temporal quantifier ‘whenever’ on our ‘shift and refer’ strategy as amounting to ‘throwing away the refer’, and changing the shift from an ‘existential shift’ to a ‘universal shift’. Note that the syntactic representation is actually somewhat simpler than in the referential cases: on the other hand, viewed as an instruction (say to examine a temporal database) the above representation is more complex — it could trigger a huge search. Actually, looking at matters this way suggests that there is an implicit reference even in the above sentence: clearly not all times are relevant to its truth. This suggests we ought to augment our representations with an explicit relativisation to relevant times. Accordingly, we’ll introduce new interval nominals, parameterised on discourses D , to remind ourselves that we’re not in general talking about all times, but only the discourse relevant ones. The ‘relevantly referring’ representation of the above bound variable sentence is thus:

$$H(\text{relevant}_D \wedge \text{Mary write a letter} \rightarrow \text{Sam asleep}).$$

(Of course, calculating relevant_D for any discourse D is exceedingly difficult. For the present discussion our new interval nominals are merely going to serve as a sort of formal semantical *momento mori*.)

One of Partee’s candidates of temporal donkeyhood is “If Mary phoned on a Friday, it was (always) Peter that answered.” This can be represented (in \mathcal{L}_c^3) by:

$$H(\text{relevant}_D \wedge \text{Mary phone} \wedge \text{friday} \rightarrow \text{Peter answer}).$$

Partee also notes an interesting negative piece of data: the sentence “If Sheila always walks into the room, Peter always wakes up”, is clearly semantically ill-formed. Partee shows that this violates the usual DRT box embedding restrictions for her translation. As far as I can see, this sentence cannot be represented in TREF.²⁴ Intuitively, the ‘shift’ is in the wrong place, and representing this wff requires introducing variables over times and some first order quantification.

As an example of an acceptable sentence that I can’t encode into TREF, consider “When John makes a phone call, he always lights up a cigarette beforehand.” This is a nice example: we need for each phone calling event a cigarette lighting event that occurs ‘just before’. The obvious attempt is:

$$L(\text{John} \dots \text{call} \rightarrow P(\text{he light} \dots \text{cigarette beforehand})),$$

but this won’t do: it would be satisfied if John works as a telephone salesman, and had once, but only once, when young tried smoking. Partee’s DRT can’t cope with such examples either. Her box construction algorithm yields essentially

$$L(\text{John} \dots \text{call} \rightarrow H(\text{he light} \dots \text{cigarette beforehand})).$$

I’ll mention this example again near the end of the section.

This concludes the survey of Partee’s examples. To sum matters up, in TREF we are able to write down expressions which represent discourses exhibiting all the various types of phenomena that Partee discusses. Moreover, the degree of accuracy of the TREF and the DRT representations seems about the same (neither provides a more detailed picture of history than the other), and both approaches have difficulty with the same data. What are we to make of this? Is temporal DRT just TREF in disguise?

A couple of observations will make at least some of this match appear less surprising. Partee’s system is a very simple one. Following Hinrichs, she assumes that whole tenseless clauses are atomic (thus, at least for the non-temporal information, she is working in some

²⁴Of course this requires a proof. I haven’t yet tried showing this.

sort of propositional calculus), and only considers a very restricted range of possibilities of inclusion and precedence. Although the two formalisms look very different, it's clear that they're attempting to model natural language at about the same level of detail.

Moreover the two formalisms have more substantial matters in common: TREF internalises forms of two key mechanisms of DRT: discourse markers and box embedding. I think it is clear that nominals and interval nominals are something like discourse markers. (Note that we can consider nominals and interval nominals to be some kind of 'well behaved free first order variables', if we like.) With L (and G and H for directed versions) in such formations as $L(\phi \rightarrow \psi)$ we effectively have a 'box embedder' in the language. These ingredients work together nicely. We get (some of) the reference we want using our 'free variables' (crucially we get most of the simple deictic and anaphoric connections), and we achieve 'bound' or 'non-referential' readings in the most natural way possible: we don't bind — we can't — instead we simply don't refer. Even the coincidence in coverage between the two formalisms is easy to explain. The box embedding and the simple formation rules of TREF (that is, its lack of binding) have the same sort of 'geometrical complexity': operator scopes mimic box embeddings and vice versa.

Now I believe one could prove results here: given a formal specification of a system of temporal DRT similar to Partee's it should be possible to show that there were non-trivial sublanguages of TREF and DRT with the same expressive power. Such translations might prove useful to both TREF and DRT — for example the former could induce at least partial proof theories on the latter. More importantly, it would pin down precisely how far the coincidence of coverage extends. Unfortunately I have never seen such a fragment of DRT (I'm told they exist), and lacking the prerequisites for a formal comparison I'll simply state my belief that TREF does capture an important chunk of the intuitions that fire DRT. Nonetheless, I think it's also clear that temporal DRT will cope with many issues that TREF cannot. In spite of the observed overlap it's misguided to regard them as rivals; rather the best thing to do is try and develop the strengths of each. What are the strengths of TREF?

I believe the obvious answer is the right one: TREF is an intensional language, and

the \mathcal{L}_c languages are ‘doubly intensional’. Both are good at coping with contextualised aspects of discourse. The sensible thing to do seems to be to develop this ability by further enriching the (at present weak) Kaplanian notion of context. In particular, I think it would be sensible to develop a suggestion of Ewan Klein’s mentioned by Partee²⁵ for capturing the content of the ‘just’ in the ‘just after’ incrementation of reference time. Klein suggests defining t' to be just after t iff t' is after t and there is no contextually relevant t'' between t' and t . It’s the notion of contextually relevant times that beckons here; is it possible to build such a notion into our \mathcal{L}_c languages? ²⁶ At present I don’t have an account of this notion I am satisfied with, but there are two reasons why I believe it should be investigated, one practical and one methodological. In practical terms I believe that such an account might help us deal with two other issues, the present perfect and the earlier ‘cigarette lighting’ sentence. Both seem to involve a notion of relevant times. The present perfect is traditionally regarded as a kind of past tense with present relevance, and the cigarette lighting example seems to trade on the notion of always having available some ‘just past’ cigarette lighting episode. Methodologically, I think the matter is worth pursuing for the following reason. Although I took pains to emphasize that the notion of context modeled in the Kaplanian framework is minimal in the extreme, part of the reasons for my interest in this theory (and more generally, the ideas involved in the California theory of reference), is that these can be extended to give non trivial models of context. I further believe that (sorted) intensional languages may prove an elegant way to exploit the richer framework. Needless to say, this is idle talk without concrete evidence. Attempting to build in a notion of contextually relevant times into the Kaplanian frameworks, and exploiting them (at least partially) by new sorts, seems an excellent way to test the worth of this program.

This concludes my case for the worth of *referential* sorting — however I hope that the

²⁵See [75, page 283, footnote 28]. Also note [75, page 284, footnote 35].

²⁶For a discussion of some logical issues relevant to such an attempt (but springing from the theory of comparatives) see [5, pages 11–17 and pages 115–118].

case for *sorting* is hardly yet begun. I feel the real worth of sorted intensional languages may lie in providing more sophisticated event logics, based around the type of information to be found in verb classifications: however that's another story, and apart from some hints mentioned in the next chapter, most of the plot is as yet unknown. I'll close here with a 'to be continued'.

Chapter 8

Two Loose Ends Tidied

We complete the thesis by attending to two outstanding issues. Although with the aid of TREF interval nominals we have some grip on interval structure, it is natural wonder what sorted interval based languages would be like. In the first section we present some preliminary results for such languages, and discuss the role of sorting in the richer framework and its possible relevance for natural language semantics. In the following section we survey the original work of Arthur Prior and Robert Bull on systems related to NTL.

8.1 Sorted interval based languages

The main aim of this section is to present the results of an initial examination of interval based languages with referential sorts.¹ We present a sequence of three such languages, culminating in the language IREF, an interval based correlate of languages of TREF, and discuss some of their properties. Actually we do a little more. When we define our languages we take the opportunity to introduce a sortal distinction on our information

¹For accounts of interval structures and interval based languages respectively the reader is referred to [5, Chapters 1.3 and 2.3] respectively.

bearing sorts, or variables. That is, as well as introducing referential sorts we are going to have two sorts of variables. The distinction imposed is rather trivial, but it does eventually lead to a brief discussion of more general issues involving sorting in interval based languages. We postpone the informal discussion till the end of the section.

We first introduce a sublanguage of IREF called simply \mathcal{L} . \mathcal{L} has three sorts: $SA = \{p, q, r, \dots\}$, $O = \{o_1, o_2, o_3, \dots\}$, and $INOM = \{e, d, c, \dots\}$. Both SA and O are information bearing sorts: SA variables represent states and activities, and O variables all other types of information. $INOM$ is a referential sort, and we call its elements *interval nominals* — though as we shall see, their properties differ from those of TREF interval nominals. The wffs of \mathcal{L} are constructed in the usual way using boolean and Priorean operators, and two new one place sentential operators, \downarrow and \uparrow . \downarrow and \uparrow are the existential operators that will deal with the subinterval relation. Their respective duals are \Downarrow and \Uparrow .²

The semantics of \mathcal{L} is grounded in *biframes* $\mathbf{T} = \langle T, <, \sqsubseteq \rangle$. Although arbitrary biframes are not interval structures — we shortly define this special class of biframes — for convenience we generally refer to the elements of any biframe as *intervals*.

By an \mathcal{L} *valuation* on a biframe $\langle T, <, \sqsubseteq \rangle$ is meant a function $V : ATOM \rightarrow Pow(T)$ that satisfies two constraints: for all atoms $e \in INOM$, $V(e)$ must be a singleton subset of T ; and for all $p \in SA$, $V(p)$ must be *downwards persistent*. This means that for all $t, t' \in T$, $t \in V(p)$ and $t' \sqsubseteq t$ implies $t' \in V(p)$. A pair $\mathbf{M} = \langle \mathbf{T}, V \rangle$ where \mathbf{T} is a biframe and V a valuation for \mathcal{L} on \mathbf{T} is called a *model*. The truth of a wff ϕ at an interval t in a model \mathbf{M} is given the usual inductive definition, augmented by the two clauses:

$$\begin{aligned} \mathbf{M} \models \downarrow \phi[t] & \quad \text{iff} \quad \exists t' (t' \sqsubseteq t \text{ and } \mathbf{M} \models \phi[t']) \\ \mathbf{M} \models \uparrow \phi[t] & \quad \text{iff} \quad \exists t' (t \sqsubseteq t' \text{ and } \mathbf{M} \models \phi[t']). \end{aligned}$$

In short, we have two tense logics running in parallel: one in FP and one in $\downarrow\uparrow$. We say that ϕ is *valid on a biframe* \mathbf{T} iff for all valuations V on \mathbf{T} and all $t \in T$, $\langle \mathbf{T}, V \rangle \models \phi[t]$;

²This perspicuous notation is due to Jaap van der Does [23].

and when this is the case we write $\mathbf{T} \models \phi$. We say ϕ *defines* a class of bframes \mathcal{T} just in case $\mathbf{T} \models \phi$ iff $\mathbf{T} \in \mathcal{T}$.

Because our interests are temporal, a special class of bframes is of particular concern: *interval structures*. These are bframes \mathbf{T} satisfying the following three clauses:

1. $<$ is an SPO.
2. \sqsubseteq is a PO.
3. For all $s, t, t' \in T$

$$s \sqsubseteq t \text{ and } t < t' \text{ implies } s < t', \text{ and}$$

$$s \sqsubseteq t \text{ and } t' < t \text{ implies } t' < s.$$

(The conditions governing the relationship between $<$ and \sqsubseteq specified in clause 3 are called right monotonicity and left monotonicity respectively.) For reasons which will become apparent later, at present I cannot axiomatise the class of interval structures. I can, however, give a ‘next best’ result: an axiomatisation of the class of bframes that satisfies all the clauses in the definition of interval structures save that $<$ need not be irreflexive. This class I call the class of *semi interval structures*. Bowing to the demands of present tractability, for the purposes of this section I will say that a wff ϕ is *valid* iff it is valid on all semi interval structures, and in such a case will write $\models \phi$.

First some model theoretic observations. Let’s examine the purely interval nominal fragment of \mathcal{L} . The basic correspondence for this part of \mathcal{L} is with a fragment of the first order language \mathbf{L}_0^\sqsubseteq . This language is \mathbf{L}_0 augmented by a second binary predicate \sqsubseteq . To translate purely interval nominal wffs into \mathbf{L}_0^\sqsubseteq we take as our base clause $ST(e) = (e = t_0)$ — interval nominals, like ordinary NTL nominals, correspond to first order variables. We use the standard translation for the boolean connectives and Priorean operators, and add the obvious clauses for $\downarrow \phi$ and $\uparrow \phi$. (These mirror the clauses for P and F respectively, but introduce occurrences of \sqsubseteq instead of $<$.) It is clear that the only conditions a purely interval nominal wff can define are \mathbf{L}_0^\sqsubseteq expressible.

We can define the class of interval structures using a purely interval nominal wff. The clause

$$(FFE \rightarrow Fe) \wedge (e \rightarrow \neg Fe)$$

ensures that $<$ is a SPO;

$$(\Downarrow e \rightarrow \Downarrow e) \wedge (e \rightarrow \Downarrow (\Downarrow e \rightarrow e)) \wedge (e \rightarrow \Downarrow e)$$

ensures that \sqsubseteq is a PO; while

$$(Fe \rightarrow \Downarrow Fe) \wedge (Pe \rightarrow \Downarrow Pe)$$

implies that $<$ and \sqsubseteq are yoked together right and left monotonically. Apart from the final condition, there is nothing we haven't already seen in another guise in NTL. The only point worth remarking on is the difference in definitional powers these reveal between the interval nominals of \mathcal{L} and those of TREF. The latter can be true at a collection of indices, and cannot define either irreflexivity or antisymmetry; the former are true at a single index and can define both.

There are conditions on biframe of special interest in interval logic which are definable with the aid of interval nominals, but not in ordinary interval based languages.³ The first is *convexity*. In the intervallic setting this means that 'intervals must be unbroken' and the relevant L_0^\sqsubseteq wff is:

$$\forall xyzw((x \sqsubseteq u \wedge x < y < z \wedge z \sqsubseteq u) \rightarrow y \sqsubseteq u) \quad \text{CONV}$$

This is defined by the \mathcal{L} wff $\Downarrow (e \wedge F(d \wedge Fc)) \wedge \Downarrow c \rightarrow \Downarrow d$. A second is *descent*, expressing the option that intervals 'descend forever'. In L_0^\sqsubseteq this is expressed as

$$\forall x \forall y \sqsubseteq x \exists z \sqsubseteq y \ z \neq y \quad \text{DESC}$$

which can be defined by the \mathcal{L} wff $\Downarrow (e \rightarrow \Downarrow \neg e)$. On transitive frames this condition simplifies to $\forall x \exists y \sqsubseteq xy \neq x$, and under this assumption DESC can be defined by $e \rightarrow \Downarrow \neg e$. Another desirable property is *separability*. In the L_0^\sqsubseteq wff expressing this condition we use the predicate xOy — x overlaps y — which is defined by $\exists u(u \sqsubseteq x \wedge u \sqsubseteq y)$. Then separability is:

$$\forall xy(x < y \rightarrow \neg xOy) \quad \text{SEP}$$

³The definition of these conditions is taken from [5, pages 59 – 69]. A discussion of the correspondence theory of standard interval based languages may be found in [5, pages 202 – 209].

which can be defined in \mathcal{L} by $Fe \rightarrow \neg \downarrow \uparrow e$. The intervallic analog of *linearity* is that given two intervals, either one precedes the other or they overlap. In \mathbf{L}_0^\sqsubseteq ,

$$\forall xy(x < y \vee y < x \vee xOy) \quad \text{LIN}$$

which can be defined by the \mathcal{L} wff $Pe \vee Fe \vee \downarrow \uparrow e$. Another useful condition, also definable in \mathcal{L} , is *freedom* — “non-inclusion implies having a disjoint sub-period” [5, page 62]. In contraposed form this amounts to the \mathbf{L}_0^\sqsubseteq wff

$$\forall xy(\forall z(z \sqsubseteq x \rightarrow zOy) \rightarrow x \sqsubseteq y) \quad \text{FREE}$$

which is defined by $\downarrow \downarrow \uparrow e \rightarrow \uparrow e$.

Note the utility of the $\downarrow \uparrow$ construction in these formulas. Often the \uparrow operator is omitted from interval based languages — but clearly it is useful in collaboration with interval nominals. Such examples also make more pressing the search for criterion distinguishing conditions definable in nominal modal (\downarrow only) languages and nominal tensed (both \downarrow and \uparrow) languages.

Let’s turn to the effects of the constraint on the interpretation of *SA* variables. How far does downwards persistence ‘spread’ from the *SA* variables to complex wffs? One type of answer, involving truth in a model, is provided by van Benthem [5, pages 197 – 198]. In the terms of this chapter, what van Benthem shows is that if ϕ is any formula whose only atoms are *SA* variables, and whose only connectives are \downarrow , F , P , \wedge and \vee — negations are not allowed — then the truth of ϕ at an interval t in any interval model \mathbf{M} guarantees its truth at all subintervals t' of t in \mathbf{M} — provided that $<$ and \sqsubseteq are right and left monotonically related, and \sqsubseteq is transitive.

A second type of answer, involving validity, is provided by the following. Let ϕ be any formula of our language constructed from *any* atoms using only F , P , \wedge , \vee and \neg . We call such a wff an *FP* formula. Intuitively the constraint on the assignments to *SA* variables has no effect on which *FP* formulas are valid — this fragment cannot see its feet. We make this precise as follows. Given any biframe \mathbf{T} , by an *O*-valuation for \mathcal{L} on \mathbf{T} is meant any function $V^\circ : \text{ATOM} \rightarrow \text{Pow}(T)$ that assigns singletons to all interval nominals. That is, an *O*-valuation is free to treat *SA* variables as *O* variables. We call

a pair $\langle \mathbf{T}, V^\circ \rangle$, where \mathbf{T} is a biframe and V° an O -valuation, an O -model. We define the truth of a wff ϕ at an interval t in an O -model \mathbf{M}° just as we did for models, and we say that a wff ϕ is O -valid just in case for all semi interval structures \mathbf{T} , all O -valuation V on \mathbf{T} , and all $t \in \mathbf{T}$, $\langle \mathbf{T}, V \rangle \models \phi[t]$. In such a case we write $\models_o \phi$. What we wish to show is that for all FP wffs ϕ , $\models_o \phi$ iff $\models \phi$. The left to right direction is trivial. The construction required to establish the right to left direction is provided by the next lemma.

Lemma 8.1.1 (\sqsubseteq weakening lemma) *Let $\mathbf{M} = \langle \langle T, <, \sqsubseteq \rangle, V \rangle$ be any O -model such that $\langle T, <, \sqsubseteq \rangle$ is a semi interval structure. Define \sqsubseteq^W by:*

$$\sqsubseteq^W = \sqsubseteq \setminus \{ \langle t', t \rangle : t' \sqsubseteq t \text{ and, for some } p \in SA, t \in V(p) \text{ and } t' \notin V(p) \}$$

and let $\mathbf{M}^W = \langle \langle T, <, \sqsubseteq^W \rangle, V \rangle$. Then \mathbf{M}^W is a model and $\langle T, <, \sqsubseteq^W \rangle$ is a semi interval structure.

Proof:

As T , $<$ and V are unchanged we need only check that $\langle T, <, \sqsubseteq^W \rangle$ is a semi interval structure, and that V is downwards persistent on this new construct.

It is immediate that \sqsubseteq^W is reflexive and antisymmetric and interacts with $<$ in the required fashion. Transitivity is assured by the following argument. Suppose $t'' \sqsubseteq^W t'$ and $t' \sqsubseteq^W t$ but $t'' \not\sqsubseteq^W t$. As \sqsubseteq^W is a subrelation of \sqsubseteq , we must have both $t'' \sqsubseteq t'$ and $t' \sqsubseteq t$, which by the transitivity of \sqsubseteq means $t'' \sqsubseteq t$. As $t'' \not\sqsubseteq^W t$, it can only be that for some SA variable p , $t \in V(p)$ and $t'' \notin V(p)$. But this is impossible. As $t' \sqsubseteq^W t$ then as $t \in V(p)$ we also have $t' \in V(p)$. But then as $t'' \sqsubseteq^W t'$, as $t' \in V(p)$ we also have $t'' \in V(p)$ — contradiction. So $\langle T, <, \sqsubseteq^W \rangle$ is a semi interval structure.

But V is downwards persistent on this new semi interval structure, as we weakened \sqsubseteq in precisely the fashion required to ensure this. \square

We now have that for any FP formula ϕ and any interval model \mathbf{M} , $\mathbf{M} \models \phi[t]$ iff $\mathbf{M}^W \models \phi[t]$. This follows by induction on $\deg(\phi)$, because in the definition of \mathbf{M}^W we did

not change the original O -valuation or $<$, but merely altered \sqsubseteq . As a corollary we have that $\not\models_o \phi$ implies $\not\models \phi$. Thus we have established that $\models \phi$ iff $\models_h \phi$, for all FP formulas ϕ .⁴

Let us turn to axiomatics. To axiomatise the validities we start with K_t as a basis, and add the minimal tense logical apparatus for the \sqsubseteq relation. That is, we add as axioms all instances of:

$$\begin{aligned} \uparrow(\phi \rightarrow \psi) &\rightarrow (\uparrow\phi \rightarrow \uparrow\psi) & (\text{TL1}_{\sqsubseteq}) \\ \downarrow(\phi \rightarrow \psi) &\rightarrow (\downarrow\phi \rightarrow \downarrow\psi) & (\text{TL2}_{\sqsubseteq}) \\ \phi &\rightarrow \uparrow\downarrow\phi & (\text{TL3}_{\sqsubseteq}) \\ \phi &\rightarrow \downarrow\uparrow\phi & (\text{TL4}_{\sqsubseteq}) \end{aligned}$$

and the rules of inference $\vdash \phi \Rightarrow \vdash \uparrow\phi$, and $\vdash \phi \Rightarrow \vdash \downarrow\phi$.

Next, to control the interval nominals, we need some sort of intervallic NOM or SWEEP variant. So, first we define the notion of an *existential operator*: this is any unbroken, possibly mixed, sequence of F s, P s, \downarrow s and \uparrow s including the null sequence. Using E as a metavariable over existential operators, and ϵ as a metavariable over interval nominals, we add the following version of NOM_w :

$$\epsilon \wedge E(\epsilon \wedge \phi) \rightarrow \phi \quad (\text{NOM}_w^\epsilon.)$$

In order to deal with the relational constraints we have placed on $<$ and \sqsubseteq we shall need in addition all instances of:

$$\begin{aligned} FF\phi &\rightarrow F\phi & (4) \\ \downarrow\downarrow\phi &\rightarrow \downarrow\phi & (4_{\sqsubseteq}) \\ \phi &\rightarrow \downarrow\phi & (T_{\sqsubseteq}) \\ \epsilon &\rightarrow \downarrow(\downarrow\epsilon \rightarrow \epsilon) & (\text{Anti}_{\sqsubseteq}) \\ F\phi &\rightarrow \downarrow F\phi & (RMon) \\ P\phi &\rightarrow \downarrow P\phi & (LMon) \end{aligned}$$

⁴Although the statement and proof of this result are my own, and I have not seen it proved elsewhere, I would be rather surprised if it was new. The question it asks is so natural, and the proof so simple, that it is likely to have been noted before.

Finally, to deal with the downwards persistence of the SA variables we add as axioms all instances of

$$\nu \rightarrow \Downarrow \nu \quad (\text{Triv}_\nu)$$

where ν is a metavariable over SA variables. Call this axiomatisation K_{pe} .

The completeness proof will use the usual method of generating subframes of canonical Henkin frames, so first we need to define notions of generation and paths for biframes. By a *generated biframe* S of a biframe T is meant a triple $\langle S, <, \sqsubseteq \rangle$ where $S \subseteq T$ such that S is closed under $<, >, \sqsubseteq$ and \sqsupseteq ; $<_\bullet = < \cap S^2$; and $\sqsubseteq_\bullet = \sqsubseteq \cap S^2$. Given an biframe T and an interval $t \in T$, the *biframe generated by t* is the smallest generated biframe S_t of T containing t . By a *bipath* through a frame is meant a finite sequence of intervals $\langle t_1, \dots, t_n \rangle$, such that $t_m \in T$ ($1 \leq m \leq n$), $t = t_1$, $t' = t_n$, and

$$t_m < t_{m+1} \text{ or } t_{m+1} < t_m \text{ or } t_m \sqsubseteq t_{m+1} \text{ or } t_{m+1} \sqsubseteq t_m,$$

for all elements t_m in the sequence. If S_t is a biframe generated from some interval t , then there exists a bipath between any two intervals in S_t . We can now prove our completeness result. In what follows we assume the obvious definitions of K_{pe} -consistency and so on.

Theorem 8.1.1 *K_{pe} is sound and strongly complete with respect to the class of semi interval structures.*

Proof:

Soundness is immediate. To prove completeness, let $H^P = \langle H, <, \sqsubseteq \rangle$ be the canonical Henkin biframe for K_{pe} . (That is, H is the set of all K_{pe} -MCSs; $<$ is defined by $h < h'$ iff $G\phi \in h$ implies $\phi \in h'$, for all wffs ϕ ; and \sqsubseteq is defined by $h' \sqsubseteq h$ iff $\Downarrow \phi \in h$ implies $\phi \in h'$, for all wffs ϕ . Because we've included the minimal tense logical axioms for both the FP and \Downarrow operators pairs, all the usual tense logical lemmas hold for both $<$ and \sqsubseteq .) Given a K_{pe} -consistent set of sentences Σ , form Σ^∞ , and 'doubly generate' a sub-biframe of H^Σ from Σ^∞ . That is, we first generate a subframe of H^P from Σ^∞ , and use the fact that a bipath between any two intervals in this frame must exist, together with the fact that all instances of NOM_w^c are in every $h \in H$, to prove a Unique Occurrence Lemma. Then

we select a second generated sub-biframe of \mathbf{H}^P , constructed in such a way as to contain all and only the interval nominals driven out in the first generation stage. (In short, we proceed as was shown in the second minimal completeness proof for NTL given in Chapter 4.) Thus every interval nominal occurs in exactly one MCS in our doubly generated biframe, and the natural mapping satisfies the constraint placed on the interpretation of interval nominals. Further, it also satisfies the constraint demanded of assignments to the SA variables: this follows immediately from our inclusion of the instances of the Triv_v schema. Hence the natural mapping on this doubly generated biframe is an \mathcal{L} valuation and we have built a model. Call this model $\mathbf{M} = \langle \langle H_\Sigma, <_\Sigma, \sqsubseteq_\Sigma \rangle, V \rangle$.

The inclusion of the instances of 4 as axioms guarantees that $<_\Sigma$ is transitive, and the inclusion of the instances of 4_\sqsubseteq and T_\sqsubseteq guarantees that \sqsubseteq_Σ is both transitive and reflexive. Further, the inclusion of the RMon and LMon schemas ensures^{that} the two relations are correctly interrelated, as a straightforward argument shows. The only problematic part comes from the demand that \sqsubseteq_Σ be antisymmetric. In general this cannot be assured, nonetheless the following version of the Simple Cluster Lemma holds because of our inclusion of the instances of Anti_\sqsubseteq : any $h \in H^\Sigma$ containing any interval nominal is a simple \sqsubseteq_Σ -cluster. Obviously we want to lightly bulldoze \sqsubseteq_Σ , and this can be done as follows.

Form the structure $\langle \langle S, <_s, \sqsubseteq_s \rangle, V_s \rangle$ as follows. Create S and \sqsubseteq_s by lightly bulldozing the frame $\langle H^\Sigma, \sqsubseteq_\Sigma \rangle$. Let f be the bulldozing p-morphism used. Define V_s as described in Chapter 5; that is, $s \in V_s(a)$ iff $f(s) \in V(a)$, for all atoms a . Finally, $<_s$ is defined by $s <_s s'$ iff $f(s) <_\Sigma f(s')$. (In effect we've flattened \sqsubseteq_Σ to a PO and left $<_\Sigma$ alone.) In fact our new structure is a model. No $h \in H_\Sigma$ containing an interval nominal was in a proper cluster, hence no such cluster was bulldozed and V_s assigns singletons to interval nominals. Moreover V_s obeys the constraint on assignments to SA variables. For suppose $s_1 \sqsubseteq s_2$ and $s_2 \in V_s(p)$. As f is a morphism in \sqsubseteq_s , $f(s_1) \sqsubseteq f(s_2)$. By our definition of V_s , $f(s_2) \in V(p)$. But V is a valuation, thus it is downwards persistent in its assignments to SA variables, hence $f(s_1) \in V(p)$. By the definition of V_s we have that $s_1 \in V_s(p)$, our desired result. Hence V_s is a valuation and we have built a second model. By design

f is a multimodal p-morphism from $\langle S, <_s, \sqsubseteq_s \rangle$ to $\langle H_\Sigma, <_\Sigma, \sqsubseteq_\Sigma \rangle$ — in fact it's a strong homomorphism in $<_s$ — hence the new model is equivalent to \mathbf{M} .

It only remains to check that the relations on the new model satisfy all our structural requirements. By construction \sqsubseteq_s is a PO. That $<_s$ is transitive is immediate. It only remains to check that that $<_s$ and \sqsubseteq_s are right and left monotonically related. They are. Suppose $s_1 \sqsubseteq s_2$, and $s_2 <_s s_3$. As f is a morphism in both $<_s$ and \sqsubseteq_s we have that $f(s_1) \sqsubseteq_\Sigma f(s_2)$ and $f(s_2) <_\Sigma f(s_3)$. As the two relations in \mathbf{M} interact right monotonically we have $f(s_1) <_\Sigma f(s_3)$. But f is a strong homomorphism in $<_s$, hence $s_1 <_s s_3$. Thus right monotonicity holds in the new structure, and by a mirror image argument so does left monotonicity. Thus we have verified Σ on a model in the desired class and have our completeness result. \square

Why can we not strengthen this result to a completeness theorem for the class of all interval structures? Plausibly $K_{pe} + (\epsilon \rightarrow \neg F\epsilon)$ axiomatises this class; and surely we could prove this by performing a second round of (heavy) bulldozing, this time on the $<_s$ relation? Unfortunately this won't work: we certainly turn $<_s$ into a SPO, but we have no guarantee that the monotonicity conditions still hold afterwards. The problem lies with the structure of $<_s$ -clusters: while these have the monotonicity properties — these hold for the entire frame, so they certainly hold for all clusters — they needn't have the separability property. Suppose C is a $<_s$ -cluster and that s_1 and s_2 are elements of C such that $s_1 \sqsubseteq s_2$. Then as s_1 and s_2 are in C , $s_1 <_s s_2$ and $s_2 <_s s_1$, and s_1 and s_2 are not separated. The consequence is that when we bulldoze, s_1 and s_2 'infinitely oscillate' in a STO and monotonicity is lost.

I'll make some remarks about this. Firstly this problem is nothing to do with sorting — it's due to the inherent complexity of interval based languages. If we were working with an unsorted interval based language, and we dropped all the schemas containing metavariables over interval nominals and $Triv$, from the axiomatisation K_{pe} , we would have the axiomatisation van Benthem calls K_p . Obviously we could show that this axiomatisation is strongly complete with respect to the class of biframe \mathbf{T} such that \mathbf{T} satisfies all the conditions enjoyed by semi interval structures save that \sqsubseteq need not be

antisymmetric. (The details of the proof above prior to bulldozing establishes this.) We could then sharpen this result to prove completeness for K_p relative to the semi interval structures. (The unsorted language is blind to \sqsubseteq antisymmetry.) It would be natural to suspect that bulldozing $<$ would yield a sharpening result with respect to the interval structures — but even in the unsorted system we would be stymied at this point, and for the same reason: bulldozing will not work.

Now van Benthem proves precisely the first two results for K_p [5, pages 209 – 211]. (The sharpening to the semi interval structure is obtained by unravelling \sqsubseteq however, not bulldozing.) He then remarks, though gives no details, that a more complex unravelling argument will show that K_p is strongly complete with respect to the class of *period structures*. (Period structures are interval structures that satisfy the further CONV demand given in Chapter 6.) As yet, I have not been able produce such a proof, but van Benthem's remark makes me suspect that K_{pe} must be strongly complete with respect to the interval structures. Briefly, my reasons are as follows. Given that there is a way to unravel $<_s$ (or $<_{\sqsubseteq}$) clusters in such a way as to preserve the right and left monotonic interaction of $<_s$ and \sqsubseteq_s , we should be able to apply the method even in \mathcal{L} : by the intervallic analog of the Irreflexivity Lemma, points in such clusters are not assigned to interval nominals. Of course without details this is merely (informed) speculation, nonetheless van Benthem's result does suggest that the above setback is only temporary. While bulldozing may not be a subtle enough model transformation technique anymore, this need not mean that cluster analysis methods are exhausted.

On the other hand, these methods are beginning to get quite complex, and it is well worthwhile trying to find alternatives. There is at least one obvious option to pursue: adapting *COV* to the interval based setting. I am presently investigating the matter.

Let us now examine the second sublanguage en route to full blown IREF. We make \mathcal{L}^n out of \mathcal{L} by adjoining the elements of a fourth sort $NOM = \{i, j, k \dots\}$, the sort of *nominals*, to our stock of atoms. (These nominals are entirely new symbols.) As usual this is the sole syntactic change, and the wffs of \mathcal{L}^n are made from this enriched selection of atoms in the manner prescribed for \mathcal{L} wffs.

Nominals will be names of ‘points’, which in the intervallic setting can be taken to mean that they denote atoms; thus we will interpret \mathcal{L}^n on the class of atomic biframe. By an *atomic biframe* $\langle T, <, \sqsubseteq \rangle$ is meant a biframe such that

$$\forall t \in T \exists a (a \sqsubseteq t \ \& \ \forall t' \sqsubseteq a \ t' = a).$$

Such an interval a is called an atom. By an \mathcal{L}^n valuation on an atomic biframe \mathbf{T} is meant a function $V : \text{ATOM}_{\mathcal{L}^n} \rightarrow \text{Pow}(T)$, such that V satisfies all the constraints demanded of \mathcal{L} valuations, and in addition satisfies the constraint that for all $i \in \text{NOM}$, $V(i)$ is a singleton subset of T containing an atom. We define an \mathcal{L}^n model to be a pair $\langle \mathbf{T}, V \rangle$ where \mathbf{T} is an atomic biframe and V an \mathcal{L}^n valuation. The concept of truth at an interval in an \mathcal{L}^n model is defined by the same inductive definition as for \mathcal{L} , and we say that an \mathcal{L}^n wff is valid iff it is true at all intervals in all \mathcal{L}^n models $\langle \mathbf{T}, V \rangle$ where \mathbf{T} is an atomic semi interval structure.

We proceed straight to axiomatics. By the axiomatisation K_{pei} is meant the axiomatisation obtained by taking all \mathcal{L}^n instances of the K_{pe} axiom schemas, using the K_{pe} rules of inference in the new language \mathcal{L}^n , and in addition taking as axioms all \mathcal{L}^n instances of the following schemas:

$$\begin{aligned} n \wedge E(n \wedge \phi) &\rightarrow \phi && (\text{NOM}_W) \\ \Downarrow \phi &\rightarrow \uparrow \Downarrow \phi && (\text{McKinsey}_{\sqsubseteq}) \\ n &\rightarrow \Downarrow n && (\text{Atom}) \end{aligned}$$

(Here n is a metavariable over nominals, and E a metavariable over existential operators, not just existential tenses.) We now show how to turn the previous completeness proof into a proof that K_{pei} is strongly complete with respect to the atomic semi interval structures.

Doubly generate a Henkin structure $\langle \langle H_{\Sigma}, <_{\Sigma}, \sqsubseteq_{\Sigma} \rangle, V \rangle$; as in the previous proof.⁵ By previous reasoning we know that a Unique Occurrence Lemma holds for the interval

⁵Of course we are now dealing with K_{pei} -consistent sets of sentences, not just K_{pe} -consistent sets, and so on.

nominals, that the natural mapping respects the constraint demanded of assignments to SA variables, and that the generated model satisfies all the relational conditions demanded of semi interval structures save possibly that \sqsubseteq_Σ may not be antisymmetric. Before bulldozing we must check that the structure we have generated is in fact an \mathcal{L}^n model: that is, that the underlying biframe is atomic in \sqsubseteq_Σ , and each nominal denotes a unique atom.

The difficult part is showing that McKinsey's axiom, in conjunction with the transitivity schema 4_\sqsubseteq and the reflexivity schema T_\sqsubseteq , ensures that the Henkin structure is atomic in \sqsubseteq_Σ . However most of the work involved is standard: it's the content of Lemmon and Scott's 1966 completeness proof for $S4.1$ [61, section 5].⁶ In intervallic terms, they showed that the $McKinsey_\sqsubseteq$ schema and the 4_\sqsubseteq schema conspire to impose the structure

$$(\forall h)(\exists h_a)(h_a \sqsubseteq h \ \& \ (\forall h', h'')(h' \sqsubseteq h_a \ \& \ h \sqsubseteq h'' \Rightarrow h' = h'')),$$

on the canonical Henkin frame. (The variously decorated h s are variables over elements of the canonical Henkin frame.) The essence of their argument is as follows. The $McKinsey_\sqsubseteq$ schema is equivalent to the schema $\downarrow (\downarrow \phi \rightarrow \downarrow \downarrow \phi)$. From this observation, one can use the McKinsey axiom in conjunction with the 4_\sqsubseteq schema to show that K_{pei} has as theorems all instances of

$$\downarrow ((\downarrow \psi_1 \rightarrow \downarrow \downarrow \psi_1) \wedge \cdots \wedge (\downarrow \psi_k \rightarrow \downarrow \downarrow \psi_k)) \quad M_k,$$

for all $k \geq 1$. Given this one can show that the canonical Henkin frame must have the property noted above. Argue as follows. Pick an arbitrary $h \in H$. Form the set of sentences

$$\Sigma = \{\phi : \downarrow \downarrow \phi \in h\} \cup \{\downarrow \psi \rightarrow \downarrow \downarrow \psi : \psi \text{ is a wff}\}.$$

Because we have all instances of M_k as axioms, Σ is consistent. Let h_a be any MCS extending Σ . It is straightforward to show that h_a has the property demanded above.

⁶ $S4.1$ has as axioms, in addition to the minimal modal logical base K , $\Box \Diamond \phi \rightarrow \Diamond \Box \phi$ (McKinsey), and $\Diamond \Diamond \phi \rightarrow \Diamond \phi$ (4).

Given that the canonical Henkin frame has this property, it is immediate that any of its generated biframe has it too. In short, what Lemmon and Scott's argument establishes is that any point h in any generated biframe of the Henkin frame has a subinterval h_a such that h_a has a unique subinterval. But now we simply note that by our inclusion of the reflexivity schema T_{\sqsubseteq} , any such interval h_a is its own subinterval, and hence its own unique subinterval. That is, all such h_a are atoms, and our doubly generated Henkin structure is atomic in \sqsubseteq_{Σ} .

We next note that each nominal denotes a unique atom: because of the instances of NOM_W in every MCS, a Unique Occurrence Lemma can be proved for the nominals; moreover, because of the inclusion of the instances of the *Atom* schema, these unique denotations of the nominals must always be atoms. Thus our generated structure $\langle\langle H_{\Sigma}, <_{\Sigma}, \sqsubseteq_{\Sigma} \rangle, V\rangle$ is a \mathcal{L}^n model; call this model \mathbf{M} .

It only remains to show that we can turn \mathbf{M} into an equivalent model based on an atomic semi interval structure by light bulldozing. So, lightly bulldoze \sqsubseteq_{Σ} to make the structure $\langle S, V_s \rangle$ as described in the previous proof. We know that $S = \langle S, <_s, \sqsubseteq_s \rangle$ is a semi interval structure, but has bulldozing affected the assignment to the nominals, and is S atomic? First note that no atom of \mathbf{M} could possibly have been in a proper \sqsubseteq_{Σ} -cluster, so no atoms were bulldozed — hence nominals are still assigned singletons in S by V_s . (We don't yet know that they're assigned atoms however.) Secondly note that S is atomic. For let s be an arbitrary element of S . As \mathbf{M} is atomic, there is an atom h_a in \mathbf{M} such that $h_a \sqsubseteq_{\Sigma} f(s)$. I will show that there is an $a \in S$ such that $f(a) = h_a$, and a is an atom such that $a \sqsubseteq_s s$. By the bulldozing definition, at least one element $a \in S$ is such that $f(a) = h_a$. Moreover this element must be unique; this follows from the bulldozing definition by noting that h_a , being an atom, was not bulldozed. But now we have that $a' \sqsubseteq_s a$ implies $a' = a$. For suppose $a' \sqsubseteq_s a$. Then $f(a') \sqsubseteq_{\Sigma} f(a)$, that is, $f(a') \sqsubseteq_{\Sigma} h_a$. But h_a is its own only subinterval, thus $f(a') = h_a$ and by the previous remark $a' = a$. Thus a is an atom. As $h_a \sqsubseteq_{\Sigma} f(s)$, and as h_a is not in any \sqsubseteq_{Σ} -cluster, it follows by the first clause in the bulldozing definition that $a \sqsubseteq_s s$, and hence S is an atomic semi interval structure. That the unique elements that V_s assigns to nominals

are in fact atoms in \mathbf{S} is now immediate, hence we have built a verifying model on an atomic semi interval structure and have established:

Theorem 8.1.2 *K_{pei} is strongly complete with respect to the class of atomic semi interval structures.* \square

We now take the final step and define the language IREF. This is simply \mathcal{L}^n augmented by the shifter in the usual way. That is, we now have an additional syntactic clause stating that if ϕ is a wff, so is $L\phi$; and semantically L makes use of the universal relation on any biframe as its accessibility relation. To axiomatise the logic of the atomic semi interval structures (with universal relation) we simply take the apparatus of K_{pei} , modify it by changing the existential operators E in the NOM_W^ϵ and NOM_W schemas to the M operator, and then adding the following new axioms:

$$\begin{array}{ll} M\epsilon & (Force_\epsilon) \\ Mn & (Force) \\ L\phi \rightarrow G\phi \wedge H\phi \wedge \downarrow\phi \wedge \uparrow\phi & (Inc). \end{array}$$

Call this axiomatisation K_{iref} . I leave the proof that this suffices to the reader: all the important work has been done already, both in the previous two proofs, and in the work on TREF in Chapter 6.

This concludes the technical discussion. Is sorting in interval based systems likely to lead anywhere interesting, and will they be useful in natural language semantics?

Interval based frameworks offer a wide range of possibilities for sorting, far wider than our simple minded sortal constraint on the SA variables might suggest.⁷ Indeed the

⁷Simple minded for (at least) two reasons: firstly, we have lumped together two types of information which need teasing apart; secondly, even in its own terms this lumping together is probably inadequate. (Arguably we need an ‘upward closure’ constraint on SA valuations — given that a piece of SA information holds at every interval in some set, it should hold in their union, assuming that this interval exists. See [5, page 200] for discussion.)

interval framework, not the point based one, seems to be the spiritual home of temporal sorting. Certainly constraints on valuations have long been considered important in this tradition. For example Humberstone [48], in one of the original papers on interval based logics, insisted that valuations had to be downwards persistent. Another constraint, proposed by Hamblin and discussed by van Benthem [5, page 198], is that there should be no ‘indefinitely finely intermingled’ intervals of truth and falsity for propositional variables. All that is missing (at least to the best of my knowledge) is the idea of reflecting different constraints of interest syntactically and constructing ‘mixed but uniform’ calculi as suggested in this thesis.

In fact many of the matters discussed in the last section of Chapter 6, which may have seemed rather odd viewed through ‘modal logical eyes’, appear natural in the richer setting. Consider sortal incompleteness results; in the interval based setting these are both immediate and acceptable. For example, K_{pe} is biframe incomplete. To validate its axioms we would need to validate the instances of $\nu \rightarrow\!\!\downarrow\!\!\nu$. But neither $e \rightarrow\!\!\downarrow\!\!e$ nor $o \rightarrow\!\!\downarrow\!\!o$ is K_{pe} derivable, by an easy soundness argument. The sheer folly of attempting to cash the content of $\nu \rightarrow\!\!\downarrow\!\!\nu$ in terms of biframe structure requires little comment: the attempt obliterates the very distinction we took care to draw. More generally, the argument of that chapter that the task of constructing adequate *temporal* logics — rather than just ‘logics of the flow of time’ — involves considering constraints on information distributions becomes virtually self evident in the intervallic setting. Indeed, in interval based frameworks it’s not so much that we can successfully sort, but that we must.

In the point based setting, because we only have one dimension of variation ($<$), no matter how wildly the truth value of any wff may fluctuate we can always form some rough picture of the situation. (We can imagine a train rolling along a track, and a light on board blinking on and off intermittently, for example.) The crux of the matter is that it is easy to think of each point in a frame as being a ‘world’, or a ‘state of affairs’, and of frames as being sequences of such worlds. The worst that can happen is that for wildly implausible information distributions the individual worlds look crazy, and we cannot view the totality as causally linked. Perhaps surprisingly, this is tolerable to us: at some

level we seem prepared to countenance such collections of independent Humean worlds, and tense logic proceeds apace.

This no longer seems possible when dealing with interval logic's two dimensions ($<$ and \sqsubseteq) of variation. The important difference is that we can no longer straightforwardly equate indexes (here intervals) with worlds or states of affairs. Each index is now intended to be an unbroken stretch of time — things happen during them; states of affairs change; and our indexes now have internal contours. 'Intervals contain multiple worlds', and arbitrary information distributions may no longer 'fit' with the shape of our interval structures. By working with a richer notion of the 'shape of time' we have made it difficult to treat information distributions with Humean innocence — which, as far as the development of genuine temporal logics is concerned, is a good thing.

I believe that the interval based setting is a good one in which to logically investigate the types of distinction codified in Vendler style verb classification [108]. By working on a rich class of biframe — perhaps the \mathbf{Q} -containing ones — it should be possible to delimit interesting information bearing sorts that mirror some of the verb type distinctions familiar from the literature. Indeed, work by Alex Lascarides [58] suggests that this may be a fruitful approach. Working in her own variant of the IQ framework [87], she presented a solution to the imperfective paradox.⁸ Her solution hinges on sorting. Information is subdivided along lines suggested by Marc Moens and Mark Steedman [64][65][67][66], and it is the interpretational constraints reflecting these subdivisions that legitimate one inference and block the other. Her exploitation of sorting is rather indirect — use is made of entities called (propositional) parameters that pick out the required information — and I believe that the formal machinery used could be simplified, perhaps by using extended versions of the languages of this section; nonetheless, her work is an attractive piece of formal modeling that makes clear the underlying difference between the two

⁸"Max was running towards the station" \models "Max ran towards the station", whereas "Max was running to the station" $\not\models$ "Max ran to the station". Why?

deductions. Her work deserves further attention, and may be a good starting point for further formal semantical work on sorting and verb classification in the intervallic setting.

8.2 Earlier work by Prior and Bull

Systems of tense logic employing nominals were devised and explored by Arthur Prior and Robert Bull in the late 1960s. Prior's introduction of nominals into tense logic was motivated by leading philosophical concerns, and when discussing his work I will focus on these. Robert Bull's work is more technical, and here I shall concentrate on the possible relevance of his results to further work in NTL. It will become apparent that their concerns — or, at any rate Prior's — are rather different from mine, and I close the thesis by making this difference explicit.

I'll begin with Prior's work. First, some bibliographical details.⁹ The earliest treatment of nominals in Prior's writing is in Chapter 5 of "Past Present and Future" [79], in the section entitled "Development of the U-calculus within the theory of world states". In Appendix B of the same volume, in the section entitled "On the range of world-variables, and the interpretation of U-calculi in world-calculi" Prior examines them in more detail. In particular he discusses in detail defining a shifter in tense logical terms, using 'minimal path equations' akin to those embodied in *NOM*. However the single most important paper dealing with nominals is the slightly later "Tense Logic and the Logic of Earlier and Later" [82]. Here they are introduced as one component — a component regarded as unproblematic — of a system of tense logic embodying a high grade (a

⁹The following seems to exhaust Prior's major writings on nominals, though they occasionally occur as asides elsewhere. I have largely relied on the "Bibliography of the philosophical writings of A. N. Prior" assembled by Olav Flo, which may be found at the back of the collection "Papers in Logic and Ethics" [83, page 219–229].

‘third grade’) of ‘tense logical involvement’.¹⁰ Prior discusses the philosophical basis of this system and attempts to construct a system with a still higher, a fourth grade, of tense logical involvement. This part of the discussion involves the shifter, and Prior attempts another (more elegant) tense logical definition this operator. The discussion below largely follows this paper. In addition to these two main sources, nominals make a brief appearance in the paper “Now”[80], as was mentioned in the previous chapter. Moreover the chapter “Egocentric Logic” in “Worlds, Times and Selves”[84, pages 28 – 45] discusses using a nominal modal logic to model pronouns, though here the nominals are made out of ordinary variables using an operator called Q .¹¹ Many of Prior’s other writings bear on matters of relevance to nominals — for example, again in Appendix B of “Past, Present and Future”, in the section “The uniqueness of the time-series”, Prior further considers defining the shifter in tense logical terms.

In order to understand Prior’s motivation for introducing nominals we must understand two things: his philosophical motivation, and the UT -calculus.¹² One of the main goals of Prior’s work on tense logic was to establish the primacy of A-series or tensed talk over B-series or untensed talk. Briefly, he wanted to reduce talk of ‘earlier’ and ‘later’ to talk of ‘past’, ‘present’ and ‘future’, as he saw our conception of time as arising as a construction out of tensed facts. Now if tense logic is A series talk, B series talk is codified by the UT -calculus. This calculus is a two sorted first order language with identity that has two binary relation symbols, U and T . Both argument slots in U are filled by constants or variables over instants of time, and U is read ‘earlier than’; thus $U(a, b)$ means that a is earlier than b . (Thus this part of the UT -calculus is just L_0 , possibly augmented by constants.) The predicate T can be read as ‘true at’; its first slot

¹⁰The discussion of this paper is largely couched in this Quinean terminology. For Quine’s original discussion of degrees of modal involvement see [85]. Briefly, Quine finds high grades of modal involvement ontologically repugnant.

¹¹Robert Bull later presented an algebraic account of this system [11].

¹²Sometimes called the U -calculus.

is filled by an instant variable or constant, and its second slot by wffs of some language, here regarded as *terms*. Which language these wff-terms belong to depends on the grade of ‘tense logical involvement’ desired; for now we will suppose that tense logical wffs fill the second slot — this choice constitutes the first or lowest grade of tense logical involvement. $T(a, F(p \wedge q))$ means that it’s true at a that $p \wedge q$ will be true.

This ‘first grade’ *UT* calculus played an important technical role in early tense logic: in effect, it was a substitute for the still fledgling possible worlds semantics, and provided a sort of correspondence theory. The ‘minimal logic’ of the *UT* calculus is axiomatised by choosing some sufficient basis for first order logic and adding as additional axioms:

$$\begin{aligned} T(a, \neg p) &\leftrightarrow \neg T(a, p) \\ T(a, p \rightarrow q) &\leftrightarrow T(a, p) \rightarrow T(a, q) \\ T(a, Gp) &\leftrightarrow \forall b(U(a, b) \rightarrow T(b, p)) \\ T(a, Hp) &\leftrightarrow \forall b(U(b, a) \rightarrow T(b, p)) \end{aligned}$$

Note that these are essentially the modern truth definition for languages of tense logic with $T(a, \phi)$ replacing $\models \phi[t]$, and $U(a, b)$ replacing $t < t'$. Indeed the first completeness theorem in tense logic took the form

$$\vdash_{K_t} \phi \quad \text{iff} \quad \vdash_{UT} T(a, \phi), \text{ for all points } a;$$

and the minimal tense logic K_t was called ‘minimal’ because it corresponded in this fashion to the minimal *UT* calculus.¹³

But in addition to its technical role, the *UT* calculus provided a logical model of untensed talk. Prior had already noted that the *UT* calculus seemed more expressive than ordinary tense logic — in particular, he noted that neither irreflexivity ($\neg U(a, a)$) nor asymmetry ($U(a, b) \rightarrow \neg U(b, a)$) seemed to be ‘reflected’ in tense logic [79, page 45] — and given Prior’s philosophical position this is clearly unsatisfactory. How was a temporally adequate tensed talk to be constructed? Moreover, Prior believed the simple

¹³Prior credits Lemmon with this result, obtained in 1965; unfortunately no reference is given. A proof of essentially this result is sketched in [90, page 67].

UT calculus outlined above to be inadequate. He wanted to liberalise the set of wff-terms allowed to occupy the second slot of the T predicate, thus allowing such expressions as $T(a, (\forall a T(a, p) \rightarrow p))$ to be formed. This liberalised UT calculus is called the ‘second grade’ UT calculus, and it is this language that Prior wished to provide a tensed correlate for. He acknowledges that allowing such formations in the UT calculus “will be felt by some to be *the* step which must *not* be taken” [82, page 121], but appears himself to have regarded the step as unproblematic.

To model this second grade UT calculus a ‘third grade’ language is introduced, and this is where nominals come in. This third grade language is standard tense logic augmented by nominals, quantification over nominals, and a primitive shifter. Note the power of this calculus. We can simulate U and T :

$$\begin{aligned} Uij &=_{def} L(i \rightarrow Fj) \\ Ti\phi &=_{def} L(i \rightarrow \phi). \end{aligned}$$

It is important to note that we really are regarding nominals as variables to be quantified over, not as names. If we merely regard them as constants, we don’t have the power that Prior desires. For example, to transcribe the second grade UT calculus wff mentioned earlier we need to write down $L(i \rightarrow (\forall j L(j \rightarrow p) \rightarrow p))$.

The addition of nominals and quantification across them seems to have been regarded by Prior as a routine step. (Certainly viewed from the perspective of the UT calculus nominals are a natural addition to tense logic, and when so viewed the tendency to quantify over them becomes nearly irresistible.) He acknowledges that treating instants as propositions is unusual, but states that all instant talk is highly artificial anyway. Further, he states that considerations outwith tense logic suggest that we will need to quantify across propositions, and that quantification across nominals should not cause us concern. But one aspect of this ‘third grade’ language strikes him as unsatisfactory: the use of a defined shifter.

Prior does not consider the shifter to be a tense logical operator: it’s presence in the third grade language he regarded as an imperfection in the attempt to construct an adequate tensed talk. Accordingly, he attempts to construct a ‘fourth grade’ language

— by which he means the third grade language with the shifter reduced to tense logical terms. He begins by noting that under certain ontological assumptions — for example, the linearity of time — the shifter is definable. But this solution is unpalatable to him: it means assuming a stronger base tense logic than K_t . So he introduces the following definition: L^0p is defined to be p , and $L^{n+1}p$ is defined to be $HL^n p \wedge GL^n p$. He next introduces first order quantificational apparatus and defines the shifter L to be $\forall n L^n p$. It follows immediately by first order logic that this L is an S5 operator. Prior concludes that the UT calculus has been reduced to tense logic, discards the U and T operators, and on [82, pages 131 – 132] tabulates “the stages by which tense-logic so swells as to encompass earlier-later logic”.

I believe there are several matters that can be objected to here. For example, what is the criterion for ‘being tense logical’ that rules out the use of a primitive shifter but admits the use of the two forms of quantification Prior uses in the fourth grade language? Certainly the use of the numeric quantification seems to stretch the bounds of what is to count as *tense* logical. (It is no answer to say that we will need arithmetic eventually: Prior’s self imposed task was to demarcate the tensed.)¹⁴ However debating what is and is not tensed is not really where my interests lie, and I’ll defer till the end of the thesis making plain the differences between Prior’s position and my own.

Robert Bull explored the properties of the third grade language in the paper “An Approach to Tense logic” [12]. That is, he considers a language which is essentially NTL,

¹⁴In passing, in spite of the triumphal tone of the previously quoted sentence, Prior seems to have been in doubt as to whether this definition of L should be accepted as satisfactory. He begins the section with the words “We would reach a fourth grade of tense-logical involvement if we could give a tense logical definition of L ”, and introduces his definition with something like a caveat: “if we enlarge our symbolic apparatus a little we can give a purely tense logical definition of L ” [82, page 128]. Also, the earlier discussion of this issue in “Past, Present and Future” [79, pages 128–131] is couched in more circumspect tones. (The discussion here is very interesting, involving whether to admit the possibility that the time flow is not unique, and the ramifications of the decision on the choice of shifter.)

save that the nominals are strong nominals, these strong nominals may be quantified over, and the language contains a primitive shifter. Bull first proves a minimal completeness theorem for this system. As axioms he takes all the axioms used in K_{tref} (save those involving interval nominals), standard quantificational axioms and rules, and in addition all instances of the schema $(\exists n)n$. This is a sort of ‘converse’ to the *Force* schema Mn used in TREF: the latter says every nominal is true somewhere, the former that some nominal is true now. In short, it’s a strongness axiom. By making use of this axiom, and certain first order Henkin style ‘witnessing’ sentences, Bull is able to construct a model with a strong valuation, as his semantics requires, reasonably straightforwardly.

A natural question to ask is whether this method of employing quantificational apparatus to build strong valuations might be of use in proving completeness results for languages of NTL and related systems. Although as far as applying tense logic is concerned I do not favour quantification over variables over indices, for technical investigations it may prove a valuable tool. Is it possible to prove a (reasonably general) conservativity result for Bull’s languages over languages of NTL? If this could be done, then — at least for systems involving nominally definable classes of frames — given an NTL axiomatisation for which a completeness result was desired one would merely need to build the model using Bull’s quantificational axioms, observe that the strong valuation gave the correct relational structure (for example, irreflexivity) to the frame, and then ‘throw away’ the quantificational apparatus by appealing to the conservativity result. A pleasant idea, but at present I have little idea of how difficult it would be to prove such a conservativity result. (I certainly think that if quantification was added over the interval nominals of the previous section, such results would be difficult to come by.) But I think the question is an important one to investigate — as is any matter bearing on the building of strong valuations. Gargov, Passy, and Tinchev report a similar ‘quantified nominal’ conservativity result in [34], though the system in question is very different from Bull’s.

After proving completeness Bull turns from this system to consider the interpretation of future tense statements in future branching time frames. He notes that the ordinary operator F is probably too weak in such an ontology — $F\phi$ cannot guarantee that ϕ holds in every alternative future — and proposes introducing an operator \mathcal{F} with the

property that $\mathcal{F}\phi$ being true at t means that on every ‘course of history’ running through t , ϕ is true at some point in the future. This is not the novel part — Prior had already informally discussed such an operator — it’s the next move that’s interesting: he states that as we are concerned with ‘courses of history’ it is sensible to introduce variables over them. In effect he introduces a new referential sort, a referential sort for courses of history.¹⁵ A completeness theorem is proved for languages with course of history variables.

The final two sections are extremely interesting: he returns to the original language and considers how, given an algebraic model for them, we can build a Kripke model. He avoids the use of Stone representation and instead uses Robinson’s Enlargement theorem [91]: by taking the enlarged non-standard model of the first order theory of the algebraic model, one obtains a Kripke model. For standard tensed languages the instants of the time series are non-standard elements of this model; what is shown in section 7 is that when considering the Lindenbaum algebras of Bull’s system, instants arise as standard elements.

The ramifications of this are not clear to me, but it raises an immediate question: what happens with other sorts, and in particular interval nominals? More generally, this seems an interesting way to build models. In passing, since Bull published this paper it has become a great deal more straightforward to apply the methods of non-standard analysis. Nelson [70] has shown how to absorb the methods into set theory; detours through model theory are no longer needed.

I’ll sum up the major features of Bull’s work as follows. With his introduction of the course of history variables Bull has generalised from Prior’s original conception of

¹⁵Perhaps not quite. The emphasis, as with his discussion of nominals, is primarily on these additions as variables — things to be quantified across — not names. To put the matter another way, sorting in this thesis is primarily conceived as ‘constraining the ways that frames are decorated with information’; Bull’s conception of sorting is primarily to do with ‘what we can quantify over in our object language’. Of course the two conceptions are intimately intertwined.

nominals, and anticipated the idea of sorting — and its utility — that underlies this thesis. Moreover the two major technical contributions of the paper — the completeness theorem and the use of non-standard analysis to build models — are both potentially relevant to further work.

The historical survey completed, let's consider wider issues. The major technical difference between 'third grade' tense logic and NTL is the use of quantification across nominals. This gives the early system enormous expressive power. Note that it's more powerful than D -logic, at least as far as the countable frames are concerned.¹⁶ Firstly the D -operator can be simulated by means of

$$D\phi =_{def} (\exists i)(\neg i \wedge M(i \wedge \phi)).$$

However there are classes of countable frames that cannot be defined in D -logic but which can in third grade logic. Ron Koyman's showed by means of D logical filtration that the class of frames containing a reflexive loop is not D -logically definable [57, page 49]. In grade three tense logic we need merely write down $(\exists i)M(i \wedge Fi)$.

But this result is something of a triviality: grade three systems were designed precisely to capture the power of the UT calculus. The obvious question to ask is, is this a good way to do *tense* logic? The answer is that in terms of Prior's philosophical motives, it's a very good attempt, but that from Montagovian formal semantic perspective, it's not. I'm going to conclude the thesis by developing this answer.

Formal semantics is the business of logically modelling the way natural language works. There are two important abstractions involved with it. The first we have already met, and it arises as follows: natural languages deal with the world, and if we are to model this process we need some mathematical structures that encapsulate the real world components we consider important. Frames, models, and interval structures are all examples of such 'ontological abstractions'. But there is^a second abstraction in

¹⁶The point is that strong valuations must cover frames, and we only have a countably infinite collection of nominals.

the modelling process, one that is sometimes overlooked; we cannot deal with natural language directly, but must model it by means of artificial languages. Such abstractions will highlight some aspects of natural language and diminish others, but any abstractive process has this effect, and it is difficult to see how a science can proceed otherwise. What is important is to try and model sensitively, to try and reveal general features of the way natural language functions. The use of strong artificial languages may reveal something of the way natural language works, but if we are ever to get beyond the ‘existence proof’ stage of natural language semantics — the belief that formal semantics is possible, based on the representation of fragments of it in powerful languages — we must look for better models. In particular, we should be looking for models which tell us why we can use language so fast and efficiently.

Priorean tense logic in its original formulation conforms to the spirit of this enterprise. Let’s assume that modelling tenses by means of operators has at least some linguistic merit; what does the resulting ‘photograph of natural language’ reveal in logical terms? From the point of view of correspondence theory, something fascinating: tense logic is a curious intermediary between first and second order languages. Blind to some simple (and important) first order conditions, at the same time it can impose sophisticated conditions on the flow of time. The Priorean model is interesting because it confirms that natural language cuts the cake of expressibility along unusual lines. Priorean tense logic is sometimes damned as a distorting picture of the subtleties of natural language temporal usage. The reverse is true: Priorean tense logic is one of the few models that actually respects this subtlety, and probably the only one that has given us (via correspondence theory) any logical inkling of what this subtlety actually consists of. Tense logic can be criticised for being a blurred photograph, but Prior pointed the camera in the right direction.

In this thesis we have shown how to improve Prior’s picture. By taking into account that tense is not just about ‘shift’, but about ‘refer’ as well, we have been able to simply model a variety of natural language temporal usages — and the attempt resulted in another intermediary between first and second order logic, this time with more grip on natural temporal conditions. The point is we *modeled*. We didn’t assume we knew what

tensed talk was and then proceed to achieve it by cheerful use of first order language.

‘Tensed talk’ is that which we stalk, not that which we know.

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